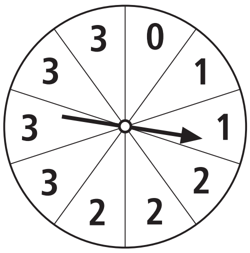
1 Discrete random variables

In this chapter you will learn

- what is meant by the probability distribution of a discrete random variable
- how to find the mean, variance and standard deviation of a discrete random variable and of a function of a discrete random variable

A Probability distribution (answers p 116)

The diagram shows a spinner. The circle is divided into 10 equal sectors. The pointer is equally likely to stop in each sector.



The probabilities of scoring 0, 1, 2, 3 are as shown in this table.

Score	0	1	2	3
Probability	0.1	0.2	0.3	0.4

The score is an example of a **discrete random variable**.

K A discrete random variable is a variable that can take individual values (usually integers), each with a given probability.

Let X stand for the score. (Capital letters are used for random variables.) $P(X = 2)$ means the probability that the score is 2. So $P(X = 2) = 0.3$.

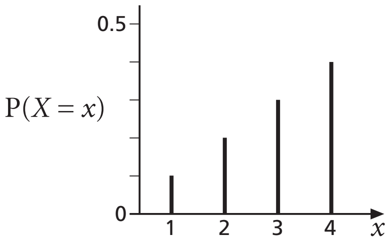
The complete set of probabilities for all the possible values of X is called the **probability distribution** of X.

x	0	1	2	3
$P(X = x)$	0.1	0.2	0.3	0.4

Notice that a lower-case (small) x is used for individual values of the random variable X.

A1 What is the sum of all the probabilities in the table? Complete the following statement.
$\sum P(X = x) = \ldots$

The probability distribution can also be shown as a 'stick graph'. The total of the heights of the sticks is 1.



A2 A board game is played with an ordinary dice. A player moves 1 square if the dice shows one, two or three, 2 squares if it shows four or five and 3 squares if it shows six.

The random variable X is the number of squares moved. Copy and complete this table of the probability distribution of X.

x	1	2	3
$P(X = x)$			

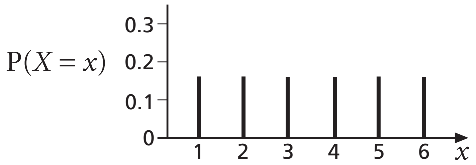
A3 In a game, two ordinary dice are thrown together. The number of squares moved is 0 if both dice show less than four, 2 if both show more than four, and 1 otherwise.

The random variable Y is the number of squares moved.
Make a table showing the values of $P(Y = y)$ for $y = 0$, 1 and 2.

A discrete **uniform** distribution is one where all the probabilities are equal.

An example is the score X from a single throw of an ordinary dice.
The distribution and its stick graph are shown below.

x	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$



This probability distribution can be summarised as

$$P(X = x) = \begin{cases} \frac{1}{6} & x = 1, 2, 3, 4, 5, 6 \\ 0 & \text{otherwise} \end{cases}$$

The '0 otherwise' statement shows that X can only take the values 1, 2, 3, 4, 5, 6.

Specifying a distribution by means of a formula

Some probability distributions may be given as formulae in terms of x.
For example, here is the distribution for scores on the spinner on the opposite page.

$$P(X = x) = \begin{cases} \dfrac{x+1}{10} & x = 0, 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

In this example, $P(X = 0) = \frac{1}{10} = 0.1$, $P(X = 1) = \frac{2}{10} = 0.2$, and so on.
You have already checked, in question A1, that $\sum P(X = x) = 1$ for this distribution.

K If X is any discrete random variable, then $\sum P(X = x) = 1$.

A4 A discrete random variable X has the probability distribution given by

$$P(X = x) = \begin{cases} \dfrac{x+3}{18} & x = 0, 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

Find the values of $P(X = x)$ for $x = 0, 1, 2, 3$ and check that $\sum P(X = x) = 1$.

A5 Explain why the function below cannot be a probability distribution.

$$P(X = x) = \begin{cases} \dfrac{x}{20} & x = 1, 2, 3, 4, 5, 6 \\ 0 & \text{otherwise} \end{cases}$$

A6 It is suggested that the following is the probability distribution of a discrete random variable X.

$$P(X = x) = \begin{cases} \dfrac{4-x}{5} & x = 1, 2, 3, 4, 5 \\ 0 & \text{otherwise} \end{cases}$$

Explain why the function cannot be a probability distribution.

Example 1

A probability distribution is given as $P(X=x)=\begin{cases} kx & x=1,2,3,4,5,6 \\ 0 & \text{otherwise}\end{cases}$

(a) Find the value of k. **(b)** Find $P(X \geq 4)$.

Solution

(a) The probability distribution is shown in this table.

x	1	2	3	4	5	6
$P(X=x)$	k	$2k$	$3k$	$4k$	$5k$	$6k$

$\sum P(X=x)$ must be equal to 1.

So $k+2k+3k+4k+5k+6k=1 \Rightarrow 21k=1 \Rightarrow k=\frac{1}{21}$

(b) $P(X \geq 4) = P(X=4, 5 \text{ or } 6) = P(X=4)+P(X=5)+P(X=6)$

$= \frac{4}{21}+\frac{5}{21}+\frac{6}{21}=\frac{15}{21}=\frac{5}{7}$

Exercise A (answers p 116)

1 The probability distribution of a discrete random variable X is defined by

$$P(X=x)=\begin{cases} \frac{1}{15}x & x=1,2,3,4,5 \\ 0 & \text{otherwise}\end{cases}$$

Find $P(X=1)$, $P(X=2)$, $P(X=3)$, $P(X=4)$ and $P(X=5)$, and show that they add up to 1.

2 A newsagent notices that no customer buys more than four newspapers or magazines and that customers are more likely to buy two or three than one or four.

A student suggests that the number bought might be modelled by a discrete random variable X with the following probability distribution.

$$P(X=x)=\begin{cases} \frac{1}{20}x(5-x) & x=1,2,3,4 \\ 0 & \text{otherwise}\end{cases}$$

Find the values of $P(X=x)$ and draw a stick graph of this distribution.

3 The number of flowers on plants of a certain species is modelled as a discrete random variable X with the probability distribution $P(X=x)$ as defined below.

$$P(X=x)=\begin{cases} kx^2 & x=1,2,3,4 \\ 0 & \text{otherwise}\end{cases}$$

(a) Write down, in terms of k, the values of $P(X=1)$, $P(X=2)$, $P(X=3)$ and $P(X=4)$.

(b) Find the value of k. **(c)** Find $P(X \geq 3)$.

4 The probability distribution of a discrete random variable X is given by

$$P(X=x)=\begin{cases} k(4-x) & x=1,2,3 \\ k(x-3) & x=4 \\ 0 & \text{otherwise}\end{cases}$$

Find the value of k.

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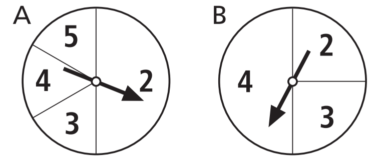
B Mean or expected value (answers p 116)

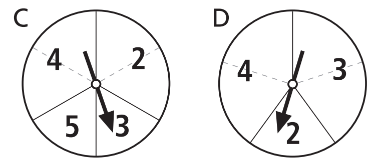
D

B1 Here are two spinners. You are invited to choose a spinner and spin it. You will win the amount the arrow points to.

Which spinner would you choose, and why?



B2 Which of these two spinners would you choose, and why?



When you decide which spinner is 'better', you need to take account of both the scores and their probabilities. A high score with a very low probability is no better than a low score with a high probability.

Here is the probability distribution of the score on spinner C.

Score	2	3	4	5
Probability	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$

One way to think about the problem is to imagine that the spinner is spun a large number of times.

For example, for spinner C the probability of a score of 2 is $\frac{1}{3}$. If the spinner is spun 60 times, then you would expect to get a score of 2 about 20 times.

The expected frequencies of the possible scores if there are 60 spins are shown in this table.

Score	2	3	4	5
Probability	20	10	20	10

These frequencies can be used to calculate a mean score:

$$\text{Mean score} = \frac{(2\times20)+(3\times10)+(4\times20)+(5\times10)}{60} = \frac{200}{60} = 3\tfrac{1}{3}$$

It was unnecessary to multiply all the probabilities by 60 and then divide by 60 at the end. The mean score can be calculated using the probabilities themselves.

$$\text{Mean score} = \left(2\times\tfrac{1}{3}\right) + \left(3\times\tfrac{1}{6}\right) + \left(4\times\tfrac{1}{3}\right) + \left(5\times\tfrac{1}{6}\right) = 3\tfrac{1}{3}$$

This mean score can be compared with the mean scores for the other spinners, to see which one is better 'on average'.

B3 Find the mean score for each of the spinners A, B and D. Which spinner is 'best'?

The mean value of a random variable X is also called the **expected value** of X, which is written $\mathrm{E}(X)$. It is also known as the **expectation** of X.

So if X is the score obtained from spinner C, then $\mathrm{E}(X) = 3\frac{1}{3}$.

To calculate $\mathrm{E}(X)$, each possible value of the random variable is multiplied by its probability, and the products are added together.

K

The possible values of X are often denoted by $x_1, x_2, x_3, \ldots$ and the corresponding probabilities by $p_1, p_2, p_3, \ldots$

Using this notation, $\mathrm{E}(X) = \sum x_i p_i$.

x	x_1	x_2	x_3	...
$\mathrm{P}(X = x)$	p_1	p_2	p_3	...

B4 Find the mean of the random variable X whose probability distribution is shown in this table.

x	0	1	2	3
$P(X = x)$	0.35	0.3	0.2	0.15

Imagine a gambling machine on which you pay £1 to play. Suppose the payout could be zero (with probability 0.7) or £2 (with probability 0.2) or £5 (with probability 0.1).

The payout in pounds X is a discrete random variable with the probability distribution shown in this table.

x	0	2	5
$P(X = x)$	0.7	0.2	0.1

B5 Find the value of $E(X)$.
What does it tell you about the machine? (Remember you pay £1 to play.)

B6 On another machine you pay 50p per game. The payout is £0 (with probability 0.75) or £1 (with probability 0.2) or £10 (with probability 0.05).

Let Y be the payout in pounds on one play of this machine.

(a) Make a table of the probability distribution of Y.

(b) Calculate $E(Y)$.

(c) From the player's point of view, is this machine better or worse than the previous one?

B7 Gemma suggests a dice game to her brother Carl.
Carl is to roll two ordinary dice.

If the two numbers are equal, Gemma will give him 20p.
If they differ by one, she will give him 5p.
If they differ by more than one, he will give Gemma 10p.

The amount that Carl could win could be –10p (10p loss), 5p or 20p.

(a) Work out the probability of each of the three outcomes (as a fraction).
Let X represent Carl's winnings in a single game.
Make a table of the probability distribution for X.

x	–10	5	20
$P(X = x)$			

(b) Find $E(X)$. What does this tell you about the game?

B8 Gemma suggest a different game. Carl is to roll three ordinary dice.

If the three numbers are all even, Gemma will give him 20p.
If the three numbers are all odd, she will give him 50p.
If the numbers are a mixture of odd and even, Carl will give her 10p.

(a) Let Y represent Carl's winnings in this game.
Make a table of the probability distribution for Y.

(b) Find $E(Y)$. What does it tell you about the game?

B9 The table shows the probability distribution of the discrete random variable W. Given that $E(W) = 1.5$, find a and b.

w	0	1	2
$P(W = w)$	0.15	a	b

Example 2

The probability distribution of a discrete random variable X is defined by

$$P(X = x) = \begin{cases} \frac{1}{15}x & x = 1, 2, 3, 4, 5 \\ 0 & \text{otherwise} \end{cases}$$

Find E(X).

Solution

The probability distribution of X is shown here.

x	1	2	3	4	5
$P(X = x)$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{5}{15}$

$$E(X) = \left(1\times\tfrac{1}{15}\right) + \left(2\times\tfrac{2}{15}\right) + \left(3\times\tfrac{3}{15}\right) + \left(4\times\tfrac{4}{15}\right) + \left(5\times\tfrac{5}{15}\right) = \tfrac{55}{15} = 3\tfrac{2}{3}$$

Example 3

The probability function of the discrete random variable X is shown in the table.
Given that $E(X) = 2.95$, find the values of a and b.

x	1	2	3	4
$P(X = x)$	0.2	a	0.25	b

Solution

The total probability must be 1, so $\quad a + b = 0.55 \quad (1)$

$E(X) = 2.95$, so $(1\times0.2) + 2a + (3\times0.25) + 4b = 2.95$

$\Rightarrow \quad 2a + 4b = 2$ or $a + 2b = 1 \quad (2)$

By subtracting (1) from (2), $b = 1 - 0.55 = 0.45$, from which $a = 0.55 - 0.45 = 0.1$

Exercise B (answers p 116)

1 The table shows the probability distribution of a discrete random variable X. Find

x	0	1	2	3
$P(X = x)$	0.08	0.15	a	0.35

(a) the value of a **(b)** E(X)

2 The probability distribution of a discrete random variable X is given by

$$P(X = x) = \begin{cases} k\left(20 - x^2\right) & x = 1, 2, 3, 4 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of k. **(b)** Find E(X).

3 The probability distribution of a discrete random variable Y is shown in the table. Given that $E(Y) = 1.4$, find the value of **(a)** β **(b)** α

y	0	1	2	3
$P(Y = y)$	α	0.3	0.4	β

4 The probability distribution of a discrete random variable W is given by this table.
Given that $E(W) = 2.8$, find the values of α and β.

w	1	2	3	4
$P(W = w)$	α	0.3	0.3	β

5 The random variable U is uniformly distributed over the values 0, 1, 2, 3, …, 9.
Find E(U).

C Expectation of a function of a discrete random variable (answers p 117)

Suppose a tetrahedral dice has the numbers 1, 2, 3, 4 on its faces. The score X from a single throw of this dice has the probability distribution shown in this table.

Value of X	1	2	3	4
Probability	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

In a game, this dice is thrown and the score doubled. The 'double score' is a random variable. Call it Y.

Then $Y = 2X$ and the probability distribution of Y is as shown in the second table.
Each value of X is doubled; the probabilities stay the same.

Value of Y	2	4	6	8
Probability	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

C1 **(a)** Find the expected value of X.

(b) Do the same for Y, where $Y = 2X$.

(c) What is the relationship between $\mathrm{E}(Y)$ and $\mathrm{E}(X)$?

C2 In a different game with the same dice, 3 is added to the score. This gives the random variable W, where $W = X + 3$.

(a) Make a table for the probability distribution of W.

(b) Find $\mathrm{E}(W)$.

(c) How are the expected values of W and X related?

C3 Suppose the dice score is doubled and then 3 is added. This gives the random variable V, where $V = 2X + 3$.

(a) Copy and complete this table for the probability distribution of V.

Value of V	5			
Probability	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

(b) Find $\mathrm{E}(V)$.

(c) What is the relationship between $\mathrm{E}(V)$ and $\mathrm{E}(X)$?

C4 Suppose the dice score is multiplied by 4 and then 5 is added. This gives the random variable U, where $U = 4X + 5$.

(a) Without first making a table for the distribution of U, write down what you think is the value of $\mathrm{E}(U)$.

(b) By making a table, check your answer to part (a).

C5 Given that a and b are numbers, what do your results in C1–C4 suggest about the value of $\mathrm{E}(aX + b)$?

K If X is a discrete random variable and a and b are numbers, then the random variable $aX + b$ is called a **linear function** of X.

$\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$

The reason $E(aX + b) = aE(X) + b$ is as follows.

(1) First suppose that each possible value $x_1, x_2, x_3, \ldots$ is multiplied by a.
To find the new mean we have to calculate $ax_1p_1 + ax_2p_2 + ax_3p_3 + \ldots$
$= a(x_1p_1 + x_2p_2 + x_3p_3 + \ldots)$
So multiplying every value by a results in the mean being multiplied by a.

(2) Now suppose that every possible value is increased by b.
The new mean is $(x_1 + b)p_1 + (x_2 + b)p_2 + (x_3 + b)p_3 + \ldots$
$= (x_1p_1 + x_2p_2 + x_3p_3 + \ldots) + b(p_1 + p_2 + p_3 + \ldots)$
Because $p_1 + p_2 + p_3 + \ldots = 1$, it follows that adding b to each value results in adding b to the mean.

(3) If we both multiply each value by a and then add b, the mean is first multiplied by a and then b is added.
So $E(aX + b) = aE(X) + b$.

C6 W is a discrete random variable for which $E(W) = 2.5$.
Find the value of

(a) $E(5W)$ **(b)** $E(5W - 2)$ **(c)** $E(-3W)$ **(d)** $E(-3W + 1)$ **(e)** $E(10 - 2W)$

Non-linear functions

Here is the probability distribution of a discrete random variable X.

x	1	2	3	4
$P(X = x)$	0.1	0.3	0.4	0.2

Another discrete random variable Y is defined in terms of X as follows:

$$Y = \frac{12}{X}$$

Y is not of the form $aX + b$, so the simple way of calculating $E(Y)$ from $E(X)$ cannot be used.

Instead, we have to find the probability distribution of Y.

If $X = 1$, then $Y = \frac{12}{1} = 12$.

If $X = 2$, then $Y = \frac{12}{2} = 6$, and so on.

The four possible values of Y, together with the corresponding probabilities, are as shown in this table.

y	12	6	4	3
$P(Y = y)$	0.1	0.3	0.4	0.2

C7 **(a)** Find $E(Y)$.

(b) Show that, although $Y = \frac{12}{X}$, $E(Y)$ is **not** equal to $\frac{12}{E(X)}$.

One function that is important in applications is the function X^2.

C8 Let X be the score on a throw of an ordinary dice. The probability distribution of X is shown here.

x	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Suppose each score is squared. This gives the random variable X^2, whose probability distribution is shown in this table.

x^2	1	4	9	16	25	36
$P(X^2 = x^2)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

(a) Find $E(X^2)$. **(b)** Find $E(X)$ and show that $E(X^2)$ is not equal to $[E(X)]^2$.

C9 The discrete random variable Y has the probability distribution given in this table.

y	1	2	3	4
$P(Y = y)$	0.1	0.3	0.4	0.2

(a) Find $E(Y^2)$. **(b)** Show that $E(Y^2) \neq [E(Y)]^2$.

If the possible values of X are $x_1, x_2, x_3, \ldots,$
then the corresponding values of X^2 are $x_1^2, x_2^2, x_3^2, \ldots$

So $E(X^2) = x_1^2 p_1 + x_2^2 p_2 + x_3^2 p_3 + \ldots$

Similarly, $E(X^3) = x_1^3 p_1 + x_2^3 p_2 + x_3^3 p_3 + \ldots$

$E(X^{-1}) = x_1^{-1} p_1 + x_2^{-1} p_2 + x_3^{-1} p_3 + \ldots$

The general rule can be stated as follows.

K If $g(X)$ is a function of the discrete random variable X, then
$E[g(X)] = g(x_1)p_1 + g(x_2)p_2 + g(x_3)p_3 + \ldots = \Sigma g(x_i)p_i$

Two other useful results follow from this expression for $E[g(X)]$.

Expectation of the sum of two functions f(*X*) + g(*X*)

If $f(X)$ and $g(X)$ are two functions of X, then

$$\begin{aligned} E[f(X) + g(X)] &= [f(x_1) + g(x_1)]p_1 + [f(x_2) + g(x_2)]p_2 + \ldots \\ &= [f(x_1)p_1 + f(x_2)p_2 + \ldots] + [g(x_1)p_1 + g(x_2)p_2 + \ldots] \\ &= E[f(X)] + E[g(X)] \end{aligned}$$

In words, the expectation of the sum of two functions of X is the sum of the expectations.

For example, $E(X^3 + X^2) = E(X^3) + E(X^2)$.

Expectation of *k* g(*X*), where *k* is a number

If $g(X)$ is a function of X, then

$$\begin{aligned} E[kg(X)] &= kg(x_1)p_1 + kg(x_2)p_2 + kg(x_3)p_3 + \ldots \\ &= k[g(x_1)p_1 + g(x_2)p_2 + g(x_3)p_3 + \ldots] \\ &= kE[g(X)] \end{aligned}$$

For example, $E(3X^2) = 3E(X^2)$.

Example 4

The probability distribution of the discrete random variable X is given by

$$P(X = x) = \begin{cases} \frac{1}{30}x & x = 4, 5, 6, 7, 8 \\ 0 & \text{otherwise} \end{cases}$$

Find **(a)** $E(X)$ **(b)** $E(3X - 2)$ **(c)** $E(10 - X)$

Solution

(a) The probability distribution of X is

x	4	5	6	7	8
$P(X = x)$	$\frac{4}{30}$	$\frac{5}{30}$	$\frac{6}{30}$	$\frac{7}{30}$	$\frac{8}{30}$

$$E(X) = 4\times\frac{4}{30} + 5\times\frac{5}{30} + 6\times\frac{6}{30} + 7\times\frac{7}{30} + 8\times\frac{8}{30} = \frac{190}{30} = \frac{19}{3} = 6\tfrac{1}{3}$$

(b) $E(3X - 2) = 3E(X) - 2 = 19 - 2 = 17$

(c) $E(10 - X) = E(-X + 10) = -E(X) + 10 = -6\tfrac{1}{3} + 10 = 3\tfrac{2}{3}$

Example 5

The probability distribution of the discrete random variable R is shown in the table.

r	1	2	3	4
$P(R = r)$	0.2	0.4	0.3	0.1

(a) Write down the probability distribution for $12R^{-1}$.

(b) Find the value of **(i)** $E(12R^{-1})$ **(ii)** $E(24R^{-1} + 5)$

Solution

(a) $12R^{-1} = \dfrac{12}{R}$. The four possible values of $12R^{-1}$ are $\frac{12}{1}, \frac{12}{2}, \frac{12}{3}, \frac{12}{4}$ or 12, 6, 4, 3.

So the probability distribution of $12R^{-1}$ is this:

Strictly speaking, this should be $P(12R^{-1} = 12r^{-1})$. →

$12r^{-1}$	12	6	4	3
Probability	0.2	0.4	0.3	0.1

(b) **(i)** $E(12R^{-1}) = 12\times0.2 + 6\times0.4 + 4\times0.3 + 3\times0.1 = 6.3$

(ii) $24R^{-1} + 5$ is a linear function of $12R^{-1}$. It is $2(12R^{-1}) + 5$.

So $E(24R^{-1} + 5) = 2\times6.3 + 5 = 17.6$

Exercise C (answers p 117)

1 X is a discrete random variable for which $E(X) = 4.25$. Find the value of

(a) $E(3X)$ **(b)** $E(3X + 4)$ **(c)** $E(-2X)$ **(d)** $E(-2X + 7)$

2 The discrete random variable S has the probability distribution given in this table. Find

s	1	2	3	4
$P(S = s)$	0.2	0.4	0.3	0.1

(a) $E(S)$

(b) $E(3S + 2)$

(c) $E(8 - 2S)$

3 The probability distribution of a discrete random variable X is given by

$$P(X = x) = \begin{cases} k(x^2 - 3) & x = 2, 3, 4 \\ 0 & \text{otherwise} \end{cases}$$

Find **(a)** the value of k **(b)** $E(X)$ **(c)** $E(6 - X)$ **(d)** $E(5X - 1)$

4 The table shows the probability distribution of the discrete random variable X.

x	0	1	2	3	4
$P(X = x)$	0.1	0.25	0.3	0.2	0.15

Find the value of

(a) $E(X)$ **(b)** $E(X^2)$ **(c)** $E(X^3)$

5 The table shows the probability distribution of the discrete random variable X.

x	1	2	3	4
$P(X = x)$	0.4	0.3	0.2	0.1

Find **(a)** $E(X^2)$ **(b)** $E(2X^2 - 1)$

6 The probability distribution of the discrete random variable X is shown in this table.

x	0	1	2	3
$P(X = x)$	0.05	0.55	0.25	0.15

Find the value of $E\left(\frac{24}{X+1}\right)$.

7 The probability distribution of the discrete random variable T is given by

$$P(T = t) = \begin{cases} \frac{t^2}{55} & t = 1, 2, 3, 4, 5 \\ 0 & \text{otherwise} \end{cases}$$

Find the mean value of $60T^{-1}$.

D Variance and standard deviation (answers p 117)

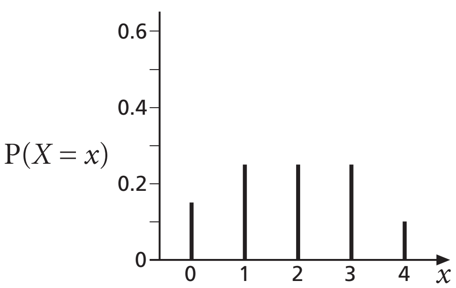
D1 The probability distributions for the scores X and Y in two different games are given below.

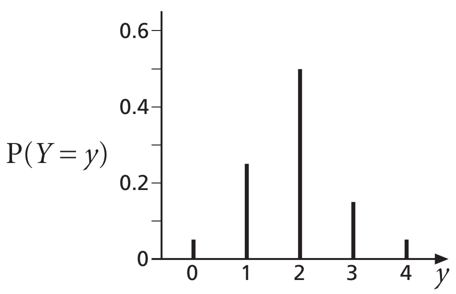
Game A

x	0	1	2	3	4
$P(X = x)$	0.15	0.25	0.25	0.25	0.10

Game B

y	0	1	2	3	4
$P(Y = y)$	0.05	0.25	0.50	0.15	0.05





(a) Show that $E(X) = E(Y)$.

(b) The distributions are similar as far as the mean score is concerned. How do they differ?

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Excerpt
More information

K The **variance** of the discrete random variable X is a measure of the spread of the distribution.

It is defined as $\sum(x_i - \mu)^2 p_i$, where μ stands for the mean value of X, or $E(X)$.

The variance of X is written as $\mathrm{Var}(X)$ and is often denoted by σ^2.

D2 For game A in question D1, $E(X) = \mu = 1.9$.

x	0	1	2	3	4
$P(X = x)$	0.15	0.25	0.25	0.25	0.10

So $\mathrm{Var}(X) = \sum(x_i - \mu)^2 p_i = (0 - 1.9)^2 \times 0.15 + (1 - 1.9)^2 \times 0.25 + \ldots$

Complete this calculation to find $\mathrm{Var}(X)$.

The definition $\sum(x_i - \mu)^2 p_i$ does not give the easiest way to calculate $\mathrm{Var}(X)$. A simpler formula can be derived from the definition, as follows.

$\mathrm{Var}(X) = \sum(x_i - \mu)^2 p_i$

$= \sum(x_i^2 - 2\mu x_i + \mu^2)p_i$ *by expanding the brackets*

$= \sum x_i^2 p_i - 2\mu \sum x_i p_i + \mu^2 \sum p_i$ *by writing the single sum as three separate sums*

$= \sum x_i^2 p_i - 2\mu^2 + \mu^2$ *because* $\sum x_i p_i = \mu$ *and* $\sum p_i = 1$

$= \sum x_i^2 p_i - \mu^2$

The expression $\sum x_i^2 p_i$ in this formula is the expression for $E(X^2)$.

So the formula may be written in a way that is easy to remember:

K $\mathrm{Var}(X) = E(X^2) - [E(X)]^2$ 'variance = expectation of square – square of expectation'

The **standard deviation** σ is defined as $\sqrt{\mathrm{Var}(X)}$.

D3 For game B in question D1, $E(Y) = 1.9$.

y	0	1	2	3	4
$P(Y = y)$	0.05	0.25	0.50	0.15	0.05

(a) Find $E(Y^2)$.

(b) Use the formula $\mathrm{Var}(Y) = E(Y^2) - [E(Y)]^2$ to find the variance of Y.

(c) Which game has the greater variance? How could you tell this from the graphs?

Example 6

Find the variance and standard deviation of the random variable T whose probability distribution is given here.

t	0	1	2	3
$P(T = t)$	0.05	0.35	0.45	0.15

Solution

$\mathrm{Var}(T) = E(T^2) - [E(T)]^2$

$E(T) = \sum t_i p_i = (0 \times 0.05) + (1 \times 0.35) + (2 \times 0.45) + (3 \times 0.15) = 1.7$

$E(T^2) = \sum t_i^2 p_i = (0^2 \times 0.05) + (1^2 \times 0.35) + (2^2 \times 0.45) + (3^2 \times 0.15) = 3.5$

So $\mathrm{Var}(T) = E(T^2) - [E(T)]^2 = 3.5 - 1.7^2 = 0.61$

Standard deviation $= \sqrt{0.61} = 0.781$ to 3 d.p.

Exercise D (answers p 118)

1 The table shows the probability distribution of a discrete random variable X. Find the value of

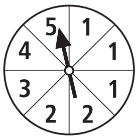
x	0	1	2	3
$P(X = x)$	0.08	0.15	0.42	0.35

(a) $E(X)$ **(b)** $Var(X)$

2 The table shows the probability distribution of a discrete random variable S.
Find the value of **(a)** $E(S)$ **(b)** $Var(S)$

x	1	2	3	4	5
$P(X = x)$	0.05	0.25	0.40	0.20	0.10

3 Find the mean and variance of the score obtained when this spinner is spun once.



4 The probability distribution of a discrete random variable X is defined by

$$P(X = x) = \begin{cases} \frac{1}{20}x & x = 2, 3, 4, 5, 6 \\ 0 & \text{otherwise} \end{cases}$$

Find $E(X)$ and $Var(X)$.

5 The random variable U has the probability distribution shown in the table. Given that $E(U) = 0.8$, find

u	0	1	2	3
$P(U = u)$	0.4	a	0.1	b

(a) the values of a and b **(b)** $Var(U)$

E Variance of a function of a discrete random variable (answers p 118)

E1 The probability distribution of a discrete random variable X is shown in this table.

x	0	1	2	3
$P(X = x)$	0.1	0.3	0.4	0.2

(a) Find Var (X).

Each possible value of X is doubled to give the random variable Y, where $Y = 2X$.

y	0	2	4	6
$P(Y = y)$	0.1	0.3	0.4	0.2

(b) **(i)** Write down the value of $E(Y)$.

(ii) Find $E(Y^2)$ and hence $Var(Y)$.

(c) Is the variance of Y double the variance of X?
If not, what is the relationship between $Var(Y)$ and $Var(X)$?

E2 The table shows the probability distribution of a discrete random variable X.

x	0	1	2	3
$P(X = x)$	0.05	0.55	0.25	0.15

(a) Find $Var(X)$.

Let $Y = 3X + 2$.

(b) Before doing any calculation, write down what you think is the value of $Var(Y)$.

(c) Copy and complete this distribution table for Y, and find $Var(Y)$.

y	2			
$P(Y = y)$	0.05	0.55	0.25	0.15

(d) How are $Var(Y)$ and $Var(X)$ related?

K The variance of a linear function $aX + b$ of the random variable X is related to the variance of X by the equation

$$\text{Var}(aX + b) = a^2\text{Var}(X)$$

The explanation is as follows.

(1) The variance, defined as $\sum(x_i - \mu)^2 p_i$, involves the squared deviations from the mean.
If every value x_i is multiplied by a, the mean is also multiplied by a.
So every deviation from the mean $(x_i - \mu)$ is also multiplied by a.
When the deviations are squared, they are each multiplied by a^2.
So the variance is multiplied by a^2.

(2) If b is added to every value x_i, it gets added to the mean as well.
So the deviations from the mean $(x_i - \mu)$ do not change and so there is no effect on the variance.

(3) If we both multiply each value by a and then add b, the variance is multiplied by a^2 but, again, adding a constant to each value has no effect on the variance.
So $\text{Var}(aX + b) = a^2\text{Var}(X)$.

E3 The table shows the probability distribution of a discrete random variable X.

x	1	2	3	4
$P(X = x)$	0.2	0.4	0.3	0.1

(a) Find $\text{Var}(X)$.

(b) Find **(i)** $\text{Var}(4X)$ **(ii)** $\text{Var}(3X - 4)$

E4 With X defined as in the previous question, let $Y = \frac{12}{X}$. This is a non-linear function.

(a) Complete this table for the distribution of Y.

y	12	6		
$P(Y = y)$	0.2	0.4	0.3	0.1

(b) Find $E(Y)$ and $E(Y^2)$.

(c) Use the formula $\text{Var}(Y) = E(Y^2) - [E(Y)]^2$ to find $\text{Var}(Y)$.

(d) Use the result of E3(a) to show that $\text{Var}\left(\frac{12}{X}\right)$ is **not** equal to $\frac{12}{\text{Var}(X)}$.

K To find the variance of a non-linear function g(X), use the formula

variance = expectation of square – square of expectation

with g(X) instead of X.

For example, $\text{Var}\left(\frac{1}{X}\right) = E\left[\left(\frac{1}{X}\right)^2\right] - \left[E\left(\frac{1}{X}\right)\right]^2$

Example 7

The mean of the score of a single throw of a normal dice is 3.5 with variance $\frac{35}{12}$.
In a game, the score is doubled and 3 subtracted.
What are the mean and variance of the score in this game?

Solution

If X is the usual score, the new score is $2X - 3$.

$E(2X - 3) = 2 \times 3.5 - 3 = 4$ $\qquad$ $\text{Var}(2X - 3) = 2^2 \times \frac{35}{12} = \frac{35}{3}$

Example 8

The probability distribution of the discrete random variable W is shown in the table.

w	0	1	2
$P(W = w)$	0.2	0.5	0.3

Find $\text{Var}(W^3)$.

Solution

You can work in terms of W^3 as given, but it is easier to use another letter, say Y, for W^3.

Let $Y = W^3$.

The probability distribution for Y is:

y	0	1	8
$P(Y = y)$	0.2	0.5	0.3

Use $\text{Var}(Y) = E(Y^2) - [E(Y)]^2$.

$E(Y) = (0\times0.2) + (1\times0.5) + (8\times0.3) = 2.9$

$E(Y^2) = (0^2\times0.2) + (1^2\times0.5) + (8^2\times0.3) = 19.7$

$\text{Var}(Y) = E(Y^2) - [E(Y)]^2 = 19.7 - 2.9^2 = 11.29$

Example 9

The discrete random variable S is such that $E(S) = 3$ and $\text{Var}(S) = 7.5$.

Find the value of **(a)** $E(S^2)$ **(b)** $E[(S + 1)^2]$

Solution

(a) Use the formula $\text{Var}(S) = E(S^2) - (E(S))^2$.

So $7.5 = E(S^2) - 32$, from which $E(S^2) = 16.5$.

(b) $E[(S + 1)^2] = E(S^2 + 2S + 1)$

$= E(S^2) + E(2S + 1)$ *using 'expectation of sum = sum of expectations'*

$= E(S^2) + 2E(S) + 1 = 16.5 + 6 + 1 = 23.5$

Exercise E (answers p 118)

1 If X is the score on a single throw of a tetrahedral dice numbered 1–4, then $E(X) = 2.5$ and $\text{Var}(X) = 1.25$. Find

(a) $E(3X)$ **(b)** $E(2X + 3)$ **(c)** $\text{Var}(2X)$ **(d)** $\text{Var}(3X + 2)$

2 X is a discrete random variable for which $E(X) = 1.5$ and $\text{Var}(X) = 0.18$. Find the value of

(a) $E(4X)$ **(b)** $\text{Var}(4X)$ **(c)** $E\left(4 - \frac{1}{3}X\right)$ **(d)** $\text{Var}\left(4 - \frac{1}{3}X\right)$

3 The probability distribution of the discrete random variable X is given by

$$P(X = x) = \begin{cases} k(x+2) & x = 0, 1, 2, 3, 4 \\ 0 & \text{otherwise} \end{cases}$$

Find **(a)** the value of k **(b)** $E(X)$ **(c)** $\text{Var}(X)$ **(d)** $\text{Var}(10 - 2X)$

4 A computer graphics program produces circles whose radius in cm is chosen at random from the numbers 1, 2, 3, 4, 5 with equal probabilities.

The discrete random variable S is the area in cm^2 of a circle produced by the program. Leaving π in your answer, find

(a) the mean value of S

(b) the variance of S

(c) the standard deviation of S

5 This table shows the probability distribution of the discrete random variable X. Find

x	1	2	3	4
$P(X = x)$	0.1	0.2	0.4	0.3

(a) $\text{Var}(X)$ **(b)** $\text{Var}(X^2)$ **(c)** $\text{Var}(24X^{-1})$

6 A discrete random variable X is such that $E(X) = 8$ and $\text{Var}(X) = 3$.
Another discrete random variable Y is defined by $Y = aX + b$.
Given that $E(Y) = 30$ and $\text{Var}(Y) = 27$, find the values of a and b.

7 T is a discrete random variable such that $E(T) = 5$, $E(T^2) = 35$ and $E(T^4) = 1250$.

(a) Show that $\text{Var}(T) = 10$.

(b) Show that $\text{Var}(T^2) = 25$.

8 The discrete random variable V is such that $E(V) = 2.5$ and $\text{Var}(V) = 4.25$.

(a) Show that $E(V^2) = 10.5$.

(b) Find the value of $E[V(V-1)]$.

9 The discrete random variable W has the probability distribution shown in this table.

w	1	2	3	4	5
$P(W = w)$	0.1	0.2	0.3	0.25	0.15

(a) Find the value of

(i) $E(W)$ **(ii)** $E(W^2)$ **(iii)** $\text{Var}(W)$

A rectangle is drawn with sides of length W and $W + 10$ units.

(b) Write down an expression for the perimeter of the rectangle.

(c) Find the mean and variance of the perimeter of the rectangle.

(d) Show that the area of the rectangle is $W^2 + 10W$.

(e) Find the mean of the area of the rectangle.

10 The probability distribution for the discrete random variable R is tabulated below.

r	1	2	3	4	5
$P(R = r)$	0.1	0.2	0.4	0.2	0.1

(a) Given that $E(R) = 3$ and $\text{Var}(R) = 1.2$, find the mean and variance of $5(2R - 1)$.

(b) **(i)** Write down the probability distribution for $\dfrac{60}{R}$.

(ii) Show that $E\left(\dfrac{60}{R}\right) = 24.2$.

(iii) Given that $E\left(\dfrac{3600}{R^2}\right) = 759.4$, determine the variance of $\dfrac{60}{R}$. AQA 2003

Key points

- A discrete random variable takes individual values (usually integers), each with a given probability.
If X is any discrete random variable, then $\sum P(X = x) = 1$. (pp 6, 7)
- The expected value, expectation or mean of a discrete random variable X is denoted by $E(X)$ or μ. It is defined as $\sum x_i p_i$, where p_i is the probability of the value x_i. (p 9)
- If $g(X)$ is a function of the discrete random variable X, then $E[g(X)] = \sum g(x_i)p_i$. (p 14)
- $E[f(X) + g(X)] = E[f(X)] + E[g(X)]$ $\quad E[kg(X)] = kE[g(X)]$ (p 14)
- The variance of X is defined by $\text{Var}(X) = \sum(x_i - \mu)^2 p_i$. It is often denoted by σ^2.
$\text{Var}(X) = E(X^2) - [E(X)]^2$ (p 17)
- $E(aX + b) = aE(X) + b$ $\quad \text{Var}(aX + b) = a^2\text{Var}(X)$ (pp 12, 19)
- To find $\text{Var}[g(X)]$, use the formula $\text{Var}(X) = E(X^2) - [E(X)]^2$ with $g(X)$ in place of X. (p 19)

Mixed questions (answers p 118)

1 The probability distribution of a discrete random variable X is shown in the table. Find

x	1	2	3	4	5
$P(X = x)$	0.1	0.2	0.3	0.3	0.1

(a) $E(X)$ **(b)** $\text{Var}(X)$ **(c)** $E(3X + 2)$ **(d)** $\text{Var}(2X - 6)$

2 The probability distribution of a discrete random variable X is given by

$$P(X = x) = \begin{cases} kx(8 - x) & x = 4, 5, 6, 7 \\ 0 & \text{otherwise} \end{cases}$$

where k is a positive constant.

(a) Find the value of k. **(b)** Calculate $E(X)$. **(c)** Calculate $\text{Var}(X)$.
(d) Find $E(3X - 2)$. **(e)** Find $\text{Var}(3X - 2)$.

3 A dice is weighted so that the probability of getting a six is 0.55, and the other numbers are equally likely.

(a) Make a table for the probability distribution of the score on a single throw of this dice.
(b) Calculate the mean and variance of the score.

4 The discrete random variable X has the probability distribution shown in the table.

x	−1	0	1	2	3
$P(X = x)$	α	0.1	0.2	β	0.2

Given that $E(X) = 1.65$, find the values of

(a) α and β **(b)** $E(X^2)$ **(c)** $\text{Var}(X)$ **(d)** $E(2X - 1)$ **(e)** $\text{Var}(2X + 3)$

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0521605318 - Statistics 2 for AQA
SMP
Excerpt
More information

5 The discrete random variable X has the probability distribution shown in the table.

x	−2	−1	0	1	2
$P(X = x)$	α	0.3	β	0.1	0.2

Given that $E(X) = -0.1$, find the values of

(a) α and β **(b)** $E(X^2)$ **(c)** $Var(X)$ **(d)** $E(5 - X)$ **(e)** $Var(5 - X)$

6 A gambling machine is being designed. The payouts are to be £0, £2, £5 and £20.

The probability of paying out £2 has been fixed at 0.1 and the probability of paying out £5 has been fixed at 0.05.

The other probabilities have not yet been fixed: the probability of paying out £0 is a and the probability of paying out £20 is b.

X represents the payout per game.
The probability distribution of X is shown here.

x	0	2	5	20
$P(X = x)$	a	0.1	0.05	b

(a) Find an expression for $E(X)$.

(b) The machine is fair to the player if the expected value of the payout is equal to the cost of a game. If the cost of a game is £1, what must the values of a and b be for the machine to be fair?

(c) It is decided to fix b at 0.01. The cost of a game is £1. What is the expected loss per game to the player?

7 A café owner installs two machines, A and B. On each machine a game costs £2. The payouts, £X on machine A and £Y on machine B, have the distributions below.

A

x	0	2	5	10	20
$P(X = x)$	0.70	0.10	0.10	0.09	0.01

B

y	0	2	5	10	20
$P(Y = y)$	0.82	0.05	0.05	0.02	0.06

(a) Find **(i)** $E(X)$ **(ii)** $Var(X)$ **(iii)** $E(Y)$ **(iv)** $Var(Y)$

(b) Which machine is better from the player's point of view? Give the reason for your answer.

(c) How much does a player expect to lose, per game, on each machine?

(d) How can you tell by looking at the probability functions that the variance of Y is greater than the variance of X?

8 The discrete random variable T is such that $E(T) = 5$ and $Var(T) = 25$.

(a) A rectangle has sides of length $2T$ and $(T + 5)$. Determine the mean and variance of the **perimeter** of the rectangle.

(b) **(i)** Show that $E(T^2) = 50$.

(ii) Hence determine the mean of the **area** of the rectangle. AQA 2002

9 The radius, R centimetres, of a circle in a computer-generated picture is a discrete random variable with $E(R) = 2$, $E(R^2) = 5$ and $Var(R) = 1$.

Determine, in terms of π,

(a) the mean and variance of the **circumference** of the circle

(b) the mean of the **area** of the circle AQA 2002

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SMP
Excerpt
More information

Test yourself (answers p 119)

1 The probability distribution for the number of vehicles, V, involved in each minor accident on a particular stretch of road can be modelled as follows.

v	1	2	3	4	5
$P(V = v)$	0.15	0.45	0.20	0.15	0.05

(a) Show that $E(V) = 2.5$ and $\text{Var}(V) = 1.15$.

(b) The total cost, £C, of removing all the damaged vehicles following a minor accident is given by

$$C = 30V + 25$$

Determine the mean and variance of C.

(c) The total repair cost, £R, for all the vehicles involved in a minor accident is given by

$$R = 40V^2 + 15V + 50$$

Determine the value of $E(R)$. AQA 2003

2 The discrete random variable R is such that $E(R) = 3$ and $\text{Var}(R) = 1$.

(a) Determine the mean and variance of $5(R - 1)$.

The probability distribution of R is

$$P(R = r) = \begin{cases} \dfrac{r}{10} & x = 1, 2, 3, 4 \\ 0 & \text{otherwise} \end{cases}$$

(b) Calculate the mean and variance of $12R^{-1}$. AQA 2002

3 The probability distribution for the number, R, of unwrapped sweets in a tin is given in the following table.

r	1	2	3	4	5
$P(R = r)$	0.1	0.2	0.4	0.2	0.1

(a) Show that

(i) $E(R) = 3$

(ii) $\text{Var}(R) = 1.2$

(b) The number, P, of partially wrapped sweets in a tin is given by

$$P = 3R + 4$$

Find values for $E(P)$ and $\text{Var}(P)$.

(c) The total number of sweets in a tin is 200. Sweets are either correctly wrapped, partially wrapped or unwrapped.

(i) Express C, the number of correctly wrapped sweets in a tin, in terms of R.

(ii) Hence find the mean and variance of C. AQA 2004