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0521604958 - Representations of Nilpotent Lie Groups and their Applications: Basic Theory and Examples, Part I

Lawrence J. Corwin and Frederick P. Greenleaf

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# *Representations of nilpotent Lie groups and their applications*

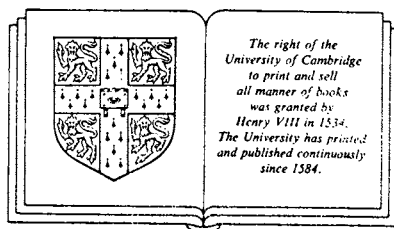
**Part I: Basic theory and examples**

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## *Preface*

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The first locally compact group whose infinite-dimensional irreducible representations were classified was a nilpotent Lie group, the Heisenberg group. The question was interesting for a quite concrete reason: understanding the Heisenberg commutation relations in quantum theory. Thus the subject of this book is one that dates back to the early days of representation theory.

Since then, both the representation theory and the applications have developed. In representation theory, the main advances were made by Dixmier in the 1950s and by Kirillov in the 1960s. Applications to ergodic theory first appear in the literature around 1960; more recently, a considerable body of literature has appeared (especially since 1975) relating partial differential operators and nilpotent Lie groups. And there are other applications to such areas as number theory.

There have been various texts describing the representation theory of nilpotent Lie groups ('Kirillov theory'), particularly Pukanszky's 1967 volume. However, there have been important new developments, and other topics not covered in these accounts have come to play an increasingly important role in applications – for example, the Moore–Wolf theory of representations with 'flat orbits' and Malcev's classic study of discrete cocompact subgroups, which offers many insights into the arithmetic theory of nilpotent groups. This book and the volume to follow attempt to give a more complete account of these topics. The current volume gives a self-contained account of the basic representation theory and harmonic analysis through the Plancherel theorem, along with a treatment of such related topics as the classification of all orbits of unipotent actions. (That this last topic is related to representation theory is one of the contributions of nilpotent theory.) We also analyze the discrete, cocompact subgroups of nilpotent Lie groups; as noted above, this can be regarded as an introduction to the arithmetic theory of the subject. The major applications of the theory, along with further topics in harmonic analysis on these groups, will be the subject of the second volume.

We believe that nilpotent Lie groups offer an excellent introduction to the subject of representation theory, and have tried to design the book accordingly. A student reading the book needs to know the elements of

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Lie theory, including the concordance between Lie groups and Lie algebras; it is also helpful to know of the existence of the Campbell–Baker–Hausdorff formula (the exact formula is less important). We also assume some knowledge of functional analysis, including, for example, the Spectral Theorem. At times we refer to such items as the Fréchet topology on  $\mathcal{S}(\mathbb{R}^n)$  or the direct integral decomposition of a representation; we have tried, however, to state exactly what facts we use and to give references, so that the reader has the option of taking the statements on faith or of looking them up. (For many topics, such as direct integral decompositions, we give quite concrete examples of the theory.)

A good number of people have encouraged and helped us in the writing of this book; we mention only a few, to keep this preface to a moderate length. Len Richardson and two of his students, Don Moss and Carolyn Pfeffer, read an earlier draft and caught many misprints, errors, and obscurities. A student of one of us, Guillaume Sanje Mpacko, contributed further suggestions. After some discussion, we have decided to conform with tradition and accept responsibility for whatever errors remain. David Tranah and the staff of the Cambridge University Press have been most helpful and accommodating. We also owe a considerable debt to Amy and Nathan Corwin, and Marjorie and Alexis Raskin, for their encouragement and their forbearance in not throwing crockery while we finished this work.