Cambridge University Press
0521604958 - Representations of Nilpotent Lie Groups and their Applications: Basic Theory and Examples, Part I
Lawrence J. Corwin and Frederick P. Greenleaf
Frontmatter
More information

CAMBRIDGE STUDIES IN ADVANCED MATHEMATICS 18

Representations of nilpotent Lie groups and their applications

Part I: Basic theory and examples

Representations of nilpotent Lie groups and their applications

Part I: Basic theory and examples

LAWRENCE J. CORWIN
Department of Mathematics
Rutgers University
New Brunswick, NJ

FREDERICK P. GREENLEAF
Department of Mathematics
New York University/Courant Institute

The right of the University of Cambridge to print and sell all manner of books was granted by Henry VIII in 1534. The University has printed and published continuously since 1584.

CAMBRIDGE UNIVERSITY PRESS
Cambridge
New York Port Chester
Melbourne Sydney

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York NY 10011–4211, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org

First published 1990
First paperback edition 2004

A catalogue record for this book is available from the British Library

Library of Congress cataloguing in publication data

Corwin, Lawrence J., 1943–
Representations of nilpotent Lie groups and
their applications/L. Corwin, F.P. Greenleaf.
p. cm. – (Cambridge studies in
advanced mathematics; 18)
Bibliography: v. 1, p.
Includes indexes.
Contents: pt. 1. Basic theory and examples.
ISBN 0 521 36034 X (v. 1) hardback
1. Lie groups, Nilpotent.
2. Representations of groups.
I. Greenleaf, Frederick P. II. Title. III. Series.
QA387.C67 1989
512′.55–dc20 89-7064 CIP

ISBN 0 521 36034 X hardback
ISBN 0 521 60495 8 paperback

Contents

Preface vii

1 Elementary theory of nilpotent Lie groups and Lie algebras 1
1.1 Basic facts about Lie groups and Lie algebras 1
1.2 Nilpotent Lie groups 10
1.3 Coadjoint orbits and the dual of $\mathfrak{g}$ 25
1.4 Notes and references 36

2 Kirillov theory 38
2.1 Some generalities on representations 38
2.2 The elements of Kirillov theory 45
2.3 Some preliminary lemmas 56
2.4 Proof of the basic theorems 62
2.5 Subgroups of codimension 1 and representations 69
2.6 Notes and references 76

3 Parametrization of coadjoint orbits 81
3.1 Orbits of unipotent actions 81
3.2 Canonical subalgebras related to elements of $\mathfrak{g}^*$ 98
3.3 The center of the enveloping algebra of a nilpotent Lie group 109
3.4 Notes and references 121

4 Plancherel formula and related topics 123
4.1 C^∞ vectors for irreducible representations of a nilpotent Lie group 123
4.2 Traces of irreducible representations 131
4.3 The canonical measures on orbits and the Plancherel formula 144
4.4 Nilpotent Lie groups that are not simply connected 161
4.5 Square-integrable representations and flat orbits 168
4.6 Representations of $\mathfrak{u}(\mathfrak{g})$ and their infinitesimal characters 184
4.7 Notes and references 190

5 Discrete cocompact subgroups **193**
5.1 Rational structures and uniform subgroups 193
5.2 Rational sugroups of nilpotent Lie groups 208
5.3 Fundamental domains for uniform subgroups 212
5.4 Lattice subgroups 214
5.5 Notes and References 224

Appendix **226**
A.1 C^∞ vectors for representations of a Lie group 226
A.2 Schwartz functions $\mathscr{S}(G)$ 235
A.3 Hermite functions on $\mathbb{R}^n$ 243

Bibliography **254**

Symbol index **261**

Subject index **263**

Preface

The first locally compact group whose infinite-dimensional irreducible representations were classified was a nilpotent Lie group, the Heisenberg group. The question was interesting for a quite concrete reason: understanding the Heisenberg commutation relations in quantum theory. Thus the subject of this book is one that dates back to the early days of representation theory.

Since then, both the representation theory and the applications have developed. In representation theory, the main advances were made by Dixmier in the 1950s and by Kirillov in the 1960s. Applications to ergodic theory first appear in the literature around 1960; more recently, a considerable body of literature has appeared (especially since 1975) relating partial differential operators and nilpotent Lie groups. And there are other applications to such areas as number theory.

There have been various texts describing the representation theory of nilpotent Lie groups ('Kirillov theory'), particularly Pukanszky's 1967 volume. However, there have been important new developments, and other topics not covered in these accounts have come to play an increasingly important role in applications – for example, the Moore–Wolf theory of representations with 'flat orbits' and Malcev's classic study of discrete cocompact subgroups, which offers many insights into the arithmetic theory of nilpotent groups. This book and the volume to follow attempt to give a more complete account of these topics. The current volume gives a self-contained account of the basic representation theory and harmonic analysis through the Plancherel theorem, along with a treatment of such related topics as the classification of all orbits of unipotent actions. (That this last topic is related to representation theory is one of the contributions of nilpotent theory.) We also analyze the discrete, cocompact subgroups of nilpotent Lie groups; as noted above, this can be regarded as an introduction to the arithmetic theory of the subject. The major applications of the theory, along with further topics in harmonic analysis on these groups, will be the subject of the second volume.

We believe that nilpotent Lie groups offer an excellent introduction to the subject of representation theory, and have tried to design the book accordingly. A student reading the book needs to know the elements of

Lie theory, including the concordance between Lie groups and Lie algebras; it is also helpful to know of the existence of the Campbell–Baker–Hausdorff formula (the exact formula is less important). We also assume some knowledge of functional analysis, including, for example, the Spectral Theorem. At times we refer to such items as the Fréchet topology on $\mathscr{S}(\mathbb{R}^n)$ or the direct integral decomposition of a representation; we have tried, however, to state exactly what facts we use and to give references, so that the reader has the option of taking the statements on faith or of looking them up. (For many topics, such as direct integral decompositions, we give quite concrete examples of the theory.)

A good number of people have encouraged and helped us in the writing of this book; we mention only a few, to keep this preface to a moderate length. Len Richardson and two of his students, Don Moss and Carolyn Pfeffer, read an earlier draft and caught many misprints, errors, and obscurities. A student of one of us, Guillaume Sanje Mpacko, contributed further suggestions. After some discussion, we have decided to conform with tradition and accept responsibility for whatever errors remain. David Tranah and the staff of the Cambridge University Press have been most helpful and accommodating. We also owe a considerable debt to Amy and Nathan Corwin, and Marjorie and Alexis Raskin, for their encouragement and their forbearance in not throwing crockery while we finished this work.