1

Elementary theory of nilpotent Lie groups and Lie algebras

1.1 Basic facts about Lie groups and Lie algebras

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$. (The results that follow hold over any field of characteristic zero, and many hold over arbitrary fields. We will be interested primarily in the fields $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$.) The *descending central series* of $\mathfrak{g}$ is defined inductively by

$$\mathfrak{g}^{(1)} = \mathfrak{g},$$
$$\mathfrak{g}^{(n+1)} = [\mathfrak{g}, \mathfrak{g}^{(n)}] = \mathbb{R}\text{-span}\{[X, Y] : X \in \mathfrak{g}, Y \in \mathfrak{g}^{(n)}\}$$

Clearly, if $\phi: \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism, then $\phi(\mathfrak{g}^{(k)}) \subseteq \mathfrak{h}^{(k)}$ for all k.

1.1.1 Lemma. *For all integers p and q, $[\mathfrak{g}^{(p)}, \mathfrak{g}^{(q)}] \subseteq \mathfrak{g}^{(p+q)}$. In particular, each $\mathfrak{g}^{(k)}$ is an ideal in $\mathfrak{g}$.*

Proof. This is clear if $p = 1$. Otherwise, we have

$$[\mathfrak{g}^{(p+1)}, \mathfrak{g}^{(q)}] = [[\mathfrak{g}, \mathfrak{g}^{(p)}], \mathfrak{g}^{(q)}] \subseteq [\mathfrak{g}^{(p)}, [\mathfrak{g}, \mathfrak{g}^{(q)}]] + [\mathfrak{g}, [\mathfrak{g}^{(p)}, \mathfrak{g}^{(q)}]]$$
$$\subseteq [\mathfrak{g}^{(p)}, \mathfrak{g}^{(q+1)}] + [\mathfrak{g}, \mathfrak{g}^{(p+q)}] = \mathfrak{g}^{(p+q+1)}$$

by Jacobi's identity and induction. □

We say that $\mathfrak{g}$ is a *nilpotent Lie algebra* if there is an integer n such that $\mathfrak{g}^{(n+1)} = (0)$. If $\mathfrak{g}^{(n)} \neq (0)$ as well, so that n is minimal, then $\mathfrak{g}$ is said to be n-step nilpotent. From Lemma 1.1.1, $\mathfrak{g}$ is n-step nilpotent if and only if all brackets of at least $n+1$ elements of $\mathfrak{g}$ are 0 but not all brackets of order n are. Note that if $\mathfrak{g}$ is nilpotent, then $\mathfrak{g}$ has a nontrivial center; in fact, if $\mathfrak{g}$ is n-step nilpotent, $\mathfrak{g}^{(n)}$ is central. Note also that $\mathbb{R}^n$, with the trivial bracket ($[X, Y] = 0$ for all X, Y) is trivially nilpotent – indeed, it is Abelian. As is obvious from the definition, these are (up to isomorphism) the only one-step nilpotent Lie algebras. Here are some other examples that we shall refer to repeatedly.

1.1.2 Example. We define $\mathfrak{h}_n$, the $(2n+1)$-dimensional *Heisenberg algebra*, to be the Lie algebra with basis $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z\}$, whose pairwise brackets are equal to zero except for

$$[X_j, Y_j] = Z, \quad 1 \leqslant j \leqslant n.$$

It is a two-step nilpotent Lie algebra. One way to realize it as a matrix algebra is to let $zZ + \Sigma_{j=1}^{n}(x_j X_j + y_j Y_j)$ correspond to the $(n+2)\times(n+2)$ matrix

$$\begin{bmatrix} 0 & x_1 & \cdots & x_n & z \\ & \cdot & & & y_1 \\ & & \cdot & & \vdots \\ & & & \cdot & y_n \\ 0 & & & & 0 \end{bmatrix}$$

1.1.3 Example. We define $\mathfrak{k}_n$ to be the $(n+1)$-dimensional Lie algebra spanned by $X, Y_1, Y_2, \ldots, Y_n$, with

$$[Y_i, Y_j] = 0,\ 1 \leqslant i, j \leqslant n,$$
$$[X, Y_j] = Y_{j+1}, 1 \leqslant j \leqslant n-1,$$
$$[X, Y_n] = 0.$$

This is an n-step nilpotent Lie algebra. One realization as a matrix algebra is obtained by letting $xX + \Sigma_{j=1}^{n} y_j Y_j$ correspond to the $(n+1)\times(n+1)$ matrix

$$\begin{bmatrix} 0 & x & 0 & \cdots & \cdots & 0 & y_n \\ & 0 & x & \cdots & \cdots & \cdots & \cdots \\ & & & \cdots & \cdots & \cdots & \cdots \\ & & & 0 & 0 & 0 & y_3 \\ & & & & 0 & x & y_2 \\ & & & & & 0 & y_1 \\ 0 & & & & & & 0 \end{bmatrix}$$

Note that $\mathfrak{k}_2 = \mathfrak{h}_1$. □

1.1.4 Example. $\mathfrak{n}_n$ is the Lie algebra of strictly upper triangular $n\times n$ matrices; it is an $(n-1)$-step nilpotent algebra, of dimension $n(n-1)/2$, and its center is one-dimensional. Note that $\mathfrak{n}_3 = \mathfrak{h}_1$. □

1.1.5 Example. $\mathfrak{f}_{n,k}$ is the 'free' nilpotent Lie algebra of step k on n generators; it is defined to be the quotient algebra $\mathfrak{f}_n/\mathfrak{f}_n^{(k+1)}$, where $\mathfrak{f}_n$ is the free Lie algebra on n generators. It is not hard to see that $\mathfrak{f}_{n,k}$ is finite-dimensional.

If $\mathfrak{g}$ is a k-step nilpotent Lie algebra generated by n elements $Y_1,\ldots,Y_n$, then $\mathfrak{g}$ is a quotient of $\mathfrak{f}_{n,k}$. For there is a homomorphism of $\mathfrak{f}_n$ onto $\mathfrak{g}$ taking the generators $X_1,\ldots,X_n$ of $\mathfrak{f}_n$ to $Y_1,\ldots,Y_n$ respectively, and $\phi(\mathfrak{f}_n^{(k+1)})=(0)$ because $\mathfrak{g}^{(k+1)}=(0)$. Hence the map factors through $\mathfrak{f}_{n,k}$. This 'universal property' of $\mathfrak{f}_{n,k}$ is sometimes useful. □

One common convention in describing nilpotent Lie algebras – and one that we shall often use – is the following. Suppose that $\mathfrak{g}=\mathbb{R}$-span $\{X_1,\ldots,X_n\}$. To describe the Lie algebra structure of $\mathfrak{g}$, it suffices to give $[X_i,X_j]$ for all i,j; in fact, we need to give $[X_i,X_j]$ only for $i<j$ (or all $i>j$). We can shorten this description considerably by giving only the nonzero brackets; all others are assumed to be zero.

Now let $\mathfrak{g}$ be any Lie algebra; let $\mathfrak{g}_{(1)}=\mathfrak{z}(\mathfrak{g})$ be the center of $\mathfrak{g}$, and define

$$\mathfrak{g}_{(j)}=\{X\in\mathfrak{g}: X \text{ is central mod } \mathfrak{g}_{(j-1)}\}=\{X\in\mathfrak{g}: [X,\mathfrak{g}]\subseteq\mathfrak{g}_{(j-1)}\}.$$

Each $\mathfrak{g}_{(j)}$ is an ideal. This is obvious if $j=1$. If $\mathfrak{g}_{(j-1)}$ is an ideal then $\mathfrak{g}_{(j-1)}\subseteq\mathfrak{g}_{(j)}$, so if $X\in\mathfrak{g}_{(j)}$ and $Y\in\mathfrak{g}$, then we have $[X,Y]\in\mathfrak{g}_{(j-1)}\subseteq\mathfrak{g}_{(j)}$, and $\mathfrak{g}_{(j)}$ is an ideal. This sequence of ideals is called the *ascending central series* for $\mathfrak{g}$.

1.1.6 Proposition. *The Lie algebra $\mathfrak{g}$ is n-step nilpotent* $\Leftrightarrow \mathfrak{g}=\mathfrak{g}_{(n)}\neq\mathfrak{g}_{(n-1)}$.

Proof. We shall prove the following two statements:

(a) if $\mathfrak{g}$ is n-step nilpotent, then $\mathfrak{g}=\mathfrak{g}_{(n)}$;
(b) if $\mathfrak{g}_{(m)}=\mathfrak{g}\neq\mathfrak{g}_{(m-1)}$, then $\mathfrak{g}^{(m+1)}=0$.

The proposition then follows: if $\mathfrak{g}$ is n-step nilpotent, (a)$\Rightarrow\exists m\leqslant n$ with $\mathfrak{g}=\mathfrak{g}_{(m)}\neq\mathfrak{g}_{(m-1)}$; then (b)$\Rightarrow\exists k\leqslant m$ such that $\mathfrak{g}$ is k-step nilpotent; but clearly $n=k$; therefore $n=m$.

We prove (a) by showing inductively that $\mathfrak{g}^{(j)}\subseteq\mathfrak{g}_{(n-j+1)}$. For $j=n$, this is clear, since $\mathfrak{g}$ is n-step nilpotent and hence $\mathfrak{g}^{(n)}$ is central. If $\mathfrak{g}^{(j+1)}\subseteq\mathfrak{g}_{(n-j)}$, then $[\mathfrak{g},\mathfrak{g}^{(j)}]\subseteq\mathfrak{g}_{(n-j)}$. Therefore the image of $\mathfrak{g}^{(j)}$ is in the center of $\mathfrak{g}/\mathfrak{g}_{(n-j)}$, so that $\mathfrak{g}^{(j)}\subseteq\mathfrak{g}_{(n-j+1)}$ and (a) follows.

For (b), we show inductively that $\mathfrak{g}^{(j)}\subseteq\mathfrak{g}_{(m+1-j)}$. Since $\mathfrak{g}_{(m)}=\mathfrak{g}^{(1)}$, the result holds for $j=1$. If it holds for j, then the image of $\mathfrak{g}^{(j)}$ is central in $\mathfrak{g}/\mathfrak{g}_{(m-j)}$. Hence $\mathfrak{g}^{(j+1)}=[\mathfrak{g},\mathfrak{g}^{(j)}]\subseteq\mathfrak{g}_{(m-j)}$, and the result holds for $j+1$. □

We now collect some easy algebraic results.

1.1.7 Lemma. *Let $\mathfrak{g}$ be a Lie algebra.*

(a) *if $\mathfrak{g}$ is nilpotent, so are all subalgebras and quotient algebras of $\mathfrak{g}$.*

(b) *The vector space sum of ideals of* $\mathfrak{g}$ *is a (not necessarily proper) ideal of* $\mathfrak{g}$.

Proof. Both parts are obvious. □

1.1.8 Lemma. Let $\mathfrak{h}$ *be a subalgebra of codimension* 1 *in a nilpotent Lie algebra* $\mathfrak{g}$. *Then* $\mathfrak{h}$ *is an ideal; in fact,* $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{h}$.

Proof. Choose any $X \notin \mathfrak{h}$; then $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}X$ as a vector space. Since $[X, X] = 0$ and $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$, it suffices to show that $[X, \mathfrak{h}] \subseteq \mathfrak{h}$. If not, we can find $Y \in \mathfrak{h}$ with $(\text{ad } Y)X = [Y, X] = \alpha X + Y_1$, $Y_1 \in \mathfrak{h}$ and $\alpha \neq 0$. By scaling Y, we may assume that $\alpha = 1$. Since $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$, induction gives

$$(\text{ad } Y)^n X = X + Y_n, \; Y_n \in \mathfrak{h}, \quad n = 1, 2, 3, \ldots .$$

If $\mathfrak{g}$ is k-step nilpotent, this gives a contradiction for $n \geqslant k$. □

The first basic theorem about nilpotent Lie algebras is classical (being roughly a century old).

1.1.9 Theorem (Engel's Theorem). *Let* $\mathfrak{g}$ *be a Lie algebra and let* $\alpha: \mathfrak{g} \to \mathfrak{gl}(V)$ *be a homomorphism such that* $\alpha(X)$ *is nilpotent for all* $X \in \mathfrak{g}$. *Then there exists a flag (Jordan–Hölder series) of subspaces*

$$(0) = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V, \quad \textit{with } \dim V_j = j,$$

such that $\alpha(X)V_j \subseteq V_{j-1}$ *for all* $j \geqslant 1$ *and all* $X \in \mathfrak{g}$. *In particular,* $\alpha(\mathfrak{g})$ *is a nilpotent Lie algebra.*

Proof. We may as well assume that $\mathfrak{g} = \alpha(\mathfrak{g}) \subseteq \mathfrak{gl}(V)$, since the theorem concerns $\alpha(\mathfrak{g})$ rather than $\mathfrak{g}$. It suffices to show that there is a one-dimensional subspace V_1 with $XV_1 = (0)$, all $X \in \mathfrak{g}$; then we get the rest of the flag by considering V/V_1. Thus we need to find $v \in V$ with $v \neq 0$ and with $Xv = 0$, all $X \in \mathfrak{g}$.

Now we use induction on $\dim \mathfrak{g}$. If $\dim \mathfrak{g} = 1$, the theorem follows directly from the nilpotence of $X \in \mathfrak{g}$. For the proof in general, we will show that $\mathfrak{g}$ contains an ideal $\mathfrak{g}_0$ of codimension 1. Assuming this for the moment, let $V_0 = \{v \in V: Xv = 0, \text{ all } X \in \mathfrak{g}_0\}$ and choose $Y \in \mathfrak{g} \sim \mathfrak{g}_0$. If we can show that $YV_0 \subseteq V_0$, we are done since the nilpotence of Y will produce $v \in V_0$ with $Yv = 0$ and $v \neq 0$. So suppose that $v_0 \in V_0$. Then for any $X \in \mathfrak{g}_0$,

$$XYv_0 = YXv_0 + [Y, X]v_0 = 0,$$

since X and $[Y, X]$ are in $\mathfrak{g}_0$. Therefore $Yv_0 \in V_0$, as claimed.

We now produce the desired ideal $\mathfrak{g}_0$. Let $\mathfrak{h}$ be any subalgebra of $\mathfrak{g}$ and let

$\mathfrak{n}(\mathfrak{h})=\{Y\in\mathfrak{g}: [Y,\mathfrak{h}]\subseteq\mathfrak{h}\}$. We will prove that if $\mathfrak{h}\neq\mathfrak{g}$, then $\mathfrak{n}(\mathfrak{h})$ properly includes $\mathfrak{h}$. Then, if $\mathfrak{g}_0$ is a proper subalgebra of maximal dimension in $\mathfrak{g}$, we have $\mathfrak{n}(\mathfrak{g}_0)=\mathfrak{g}$. If $Y\notin\mathfrak{g}_0$, then $\mathbb{R}Y+\mathfrak{g}_0$ is an algebra (as one easily checks), so that $\mathbb{R}Y+\mathfrak{g}_0=\mathfrak{g}$ by the maximality of $\mathfrak{g}_0$. Hence $\mathfrak{g}_0$ has codimension 1, as desired.

We are thus reduced to the claim about $\mathfrak{n}(\mathfrak{h})$. For this, note that if $X\in\mathfrak{g}$, then ad $X:\mathfrak{g}\to\mathfrak{g}$ is nilpotent. The reason is simple: $(\text{ad } X)Y=[X,Y]=XY-YX$, so that $(\text{ad } X)^mY$ is a linear combination of terms X^pYX^q with $p+q=m$. Assume that $X^n=0$; then $(\text{ad } X)^{2n-1}Y=0$, because either p or q is $\geq n$. Now let $\mathfrak{h}$ act on $\mathfrak{g}$ by ad. Since $(\text{ad } \mathfrak{h})\mathfrak{h}\subseteq\mathfrak{h}$, $\mathfrak{h}$ acts on $\mathfrak{g}/\mathfrak{h}$ by the quotient action $\overline{\text{ad}}$. But $\overline{\text{ad}}\, X$ is nilpotent for every $X\in\mathfrak{h}$, and $\dim\mathfrak{h}<\dim\mathfrak{g}$. By the inductive hypothesis, there exists $\bar{Y}\neq 0$ in $\mathfrak{g}/\mathfrak{h}$ such that $(\overline{\text{ad}}\, X)\bar{Y}=0$ for every $X\in\mathfrak{h}$. Now let Y be a pre-image of $\bar{Y}$. Then $Y\in\mathfrak{n}(\mathfrak{h})\sim\mathfrak{h}$, and the claim is proved. □

Note. If we take a basis for V compatible with the flag, it is clear that the matrix for each $X\in\mathfrak{g}$ is upper triangular with diagonal values equal to 0. Hence $\mathfrak{g}$ is upper triangular.

1.1.10 Corollary. *If* $\mathfrak{g}$ *is a Lie algebra such that* ad X *is nilpotent for every* $X\in\mathfrak{g}$, *then* $\mathfrak{g}$ *is nilpotent.*

Proof. The map ad: $\mathfrak{g}\to\mathfrak{gl}(\mathfrak{g})$ is a homomorphism, with kernel $\mathfrak{z}(\mathfrak{g})$.From Theorem 1.1.9, $\bar{\mathfrak{g}}=\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ is nilpotent. Suppose that $\bar{\mathfrak{g}}$ is k-step nilpotent. Then $\mathfrak{g}^{(k+1)}$ maps to 0 under the projection of $\mathfrak{g}$ on $\bar{\mathfrak{g}}$. Hence $\mathfrak{g}^{(k+1)}\subseteq\mathfrak{z}(\mathfrak{g})$, so that $\mathfrak{g}^{(k+2)}=(0)$. □

Remark. The sort of reasoning given in this last proof is necessary; it is not true that if $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ such that $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are nilpotent, then $\mathfrak{g}$ is necessarily nilpotent. The simplest example is given by $\mathfrak{g}=\mathbb{R}\text{-span}\{X,Y\}$ with $[X,Y]=X$, and $\mathfrak{h}=\mathbb{R}\text{-span}\{X\}$. More generally, if $\mathfrak{g}$ is any solvable Lie algebra, then $[\mathfrak{g},\mathfrak{g}]$ is nilpotent and $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ is abelian. □

The next theorem is a special case of Ado's Theorem – that every finite-dimensional Lie algebra over $\mathbb{C}$ is isomorphic to a subalgebra of $\mathfrak{gl}(n,\mathbb{C})$. The theorem is intrinsically interesting, but it can also be useful in calculations to have a Lie algebra realized as a subalgebra of $\mathfrak{gl}(n,\mathbb{C})$ or $\mathfrak{gl}(n,\mathbb{R})$. For one thing, if G is a Lie subgroup of $GL(n,\mathbb{R})$ – a 'linear Lie group' – then the exponential map from $\mathfrak{g}$ to G becomes the ordinary exponential map, $X\to\Sigma_{j=0}^{\infty}(X^j/j!)$; for another, the adjoint map Ad x becomes $A\to xAx^{-1}$ for $A\in\mathfrak{g}, x\in G$.

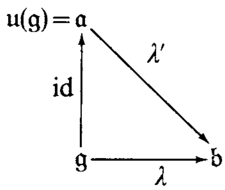
The proof uses the *universal enveloping algebra* $\mathfrak{u}(\mathfrak{g})$ associated with $\mathfrak{g}$, so we shall review a few basic facts about this construct. Given any Lie algebra $\mathfrak{g}$ we may form an associative algebra $\mathfrak{a}$ with the following properties:

(i) $\mathfrak{a}$ has a unit, contains $\mathfrak{g}$ as a vector subspace, and is generated by $\mathfrak{g}$ as an associative algebra;
(ii) $[X, Y]=XY-YX$ in $\mathfrak{a}$, for $X, Y\in\mathfrak{g}$;
(iii) If $\mathfrak{b}$ is any associative algebra with unit and $\lambda: \mathfrak{g}\to\mathfrak{b}$ is any linear map such that

$$\lambda[X, Y]=\lambda(X)\lambda(Y)-\lambda(Y)\lambda(X), \quad X, Y\in\mathfrak{g}$$

then there is a unique homomorphism $\lambda': \mathfrak{a}\to\mathfrak{b}$ such that $\lambda'(1)=1$ and $\lambda'|\mathfrak{g}=\lambda$.

In the last statement, the diagram below commutes:



The 'universal property' (iii) insures that $\mathfrak{a}$ is unique up to an isomorphism that respects the embedding $\mathfrak{g}\to\mathfrak{a}$. In general, $\mathfrak{u}(\mathfrak{g})$ may be constructed by forming the tensor algebra $T(\mathfrak{g})=\mathbb{C}1\oplus\mathbb{C}\text{-span}\{X_1\otimes\cdots\otimes X_m: X_i\in\mathfrak{g}, m<\infty\}$ and factoring out the ideal R generated by elements of the form

$$(X\otimes Y-Y\otimes X)-[X, Y], \quad \text{all } X, Y\in\mathfrak{g}.$$

If G is the simply connected Lie group associated with $\mathfrak{g}$, we can identify $\mathfrak{g}$ with the left-invariant vector fields via a map $\lambda: \mathfrak{g}\to\text{Diff}(G)$ such that $\lambda[X, Y]=\lambda(X)\lambda(Y)-\lambda(Y)\lambda(X)$. By the universal property, λ extends to a homomorphism into the algebra $\mathfrak{u}(\mathfrak{g})_L$ of left-invariant operators on G; as we will see in Chapter 3, this map is an isomorphism and gives us a different realization of $\mathfrak{u}(\mathfrak{g})$. Similarly $\mathfrak{u}(\mathfrak{g})$ is also isomorphic to the algebra $\mathfrak{u}(\mathfrak{g})_R$ of right-invariant differential operators on G. It is worth noting that we may have $\mathfrak{u}(\mathfrak{g})_L\neq\mathfrak{u}(\mathfrak{g})_R$; in fact $\mathfrak{u}(\mathfrak{g})_L\cap\mathfrak{u}(\mathfrak{g})_R$ corresponds to the center of $\mathfrak{u}(\mathfrak{g})$. Finally, there is the Poincaré–Birkhoff–Witt theorem, which states that, if $\{X_1, \ldots, X_n\}$ is any $\mathbb{R}$-basis for $\mathfrak{g}$, then the ordered monomials

$$X^\alpha=X_1^{\alpha_1}\ldots X_n^{\alpha_n}, \quad \alpha=(\alpha_1, \ldots, \alpha_n)\in\mathbb{Z}^n_+,$$

form a $\mathbb{C}$-basis for $\mathfrak{u}(\mathfrak{g})$.

With these observations in mind, we can now produce a matrix realization of any nilpotent Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ (or over $\mathbb{C}$).

1.1.11 Theorem (Birkhoff Embedding Theorem). *Let* $\mathfrak{g}$ *be a nilpotent Lie algebra over* $\mathbb{R}$. *Then there is a finite-dimensional vector space* V, *together with an injection* $i: \mathfrak{g} \to \mathfrak{gl}(V)$, *such that, for all* $X \in \mathfrak{g}$, $i(X)$ *is nilpotent.*

Proof. Assume that $\mathfrak{g}$ is k-step nilpotent; choose a basis $X_1, \dots, X_n$ of $\mathfrak{g}$ such that the first n_1 elements span $\mathfrak{g}^{(k)}$, the first n_2 span $\mathfrak{g}^{(k-1)}$, and so on. Now let us define $o(X_j) = \max\{m: X_j \in \mathfrak{g}^{(m)}\}$. The Poincaré–Birkhoff–Witt theorem states that the monomials $X_1^{\alpha_1}, \dots, X_n^{\alpha_n}$ form a basis for the enveloping algebra $\mathfrak{u}(\mathfrak{g})$. We write $X^\alpha = X_1^{\alpha_1} \dots X_n^{\alpha_n}$ and define $o(X^\alpha) = \Sigma_{j=1}^n \alpha_j o(X_j)$. Finally, set $o(0) = \infty$, $o(I) = 0$, and if $W = \Sigma_\alpha c_\alpha X^\alpha$ (with only finitely many nonzero c_α), set $o(W) = \min\{o(X^\alpha): c_\alpha \neq 0\}$.

The function o satisfies

$$\left.\begin{aligned} o(W_1 + \cdots + W_k) &\geqslant \min\{o(W_1), \dots, o(W_k)\}, \\ o(W_1 \dots W_k) &\geqslant o(W_1) + \cdots + o(W_k). \end{aligned}\right\} \text{for all } k = 1, 2, \dots. \qquad \begin{aligned}(1A)\\(1B)\end{aligned}$$

The first inequality is immediate for $k=2$, and follows by induction for larger k. Proving the second reduces to proving it when the W_j are monomials. For example, if $k=2$ and we let $X^\alpha = X_1^{\alpha_1} \dots X_n^{\alpha_n}$ for $\alpha \in \mathbb{Z}_+^n$, we have

$$\begin{aligned} o(W_1 W_2) = o\left(\sum_{\alpha \in I} c_\alpha X^\alpha \cdot \sum_{\beta \in J} d_\beta X^\beta\right) &= o\left(\sum_{\substack{\alpha \in I \\ \beta \in J}} c_\alpha d_\beta X^\alpha X^\beta\right) \\ &\geqslant \min\{o(X^\alpha X^\beta): (\alpha, \beta) \in I \times J\} \\ &\geqslant \min\{o(X^\alpha) + o(X^\beta): (\alpha, \beta) \in I \times J\} \quad \text{(valid for monomials)} \\ &\geqslant \min\{o(X^\alpha): \alpha \in I\} + \min\{o(X^\beta): \beta \in J\} = o(W_1) + o(W_2) \end{aligned}$$

The proof for $k > 2$ is the same. If $W_1 \dots W_k$ is a product of monomials, say with $W_i = X^{\alpha_i}$, where $\alpha_i = (\alpha_i(1), \dots, \alpha_i(n)) \in \mathbb{Z}_+^n$, we define its total degree to be $d = |\alpha_1| + \cdots + |\alpha_k| = \Sigma_{i,j} \alpha_i(j)$. We shall prove (1B) by induction on the total degree. If $d=1$, (1B) is trivial; so assume (1B) valid for all k and all $1 \leqslant d < N$, and consider a product whose degree is N. Let $W_i = X^{\alpha_i}$, $1 \leqslant i \leqslant k$. If $W_1 \dots W_k$ is already a monomial X^β (no transpositions required), then $\beta = (\Sigma_{i=1}^k \alpha_i(1), \dots, \Sigma_{i=1}^k \alpha_i(n))$ and

$$\begin{aligned} o(W_1 \dots W_k) = o(X^\beta) = \sum_{l=1}^n \beta_l o(X_l) &= \sum_{l=1}^n \left(\sum_{i=1}^k \alpha_i(l)\right) o(X_l) \\ &= \sum_{i=1}^k \left(\sum_{l=1}^n \alpha_i(l) o(X_l)\right) = \sum_{i=1}^k o(W_i), \end{aligned}$$

as desired. Otherwise, the product will have basis elements in the wrong order,

$$W_1 \dots W_k = X^{\alpha_1} \dots X^{\alpha_s} X^{\alpha_{s+1}} \dots X^{\alpha_k} \text{ where}$$
$$X^{\alpha_s} = W' X_i,\ X^{\alpha_{s+1}} = X_j W'', \quad \text{with } i > j.$$

If $o(X_i) = m_i$, $o(X_j) = m_j$ then $[X_i, X_j] = \Sigma_l c_l X_l$ where $c_l = 0$ unless $X_l \in \mathfrak{g}^{(m_i + m_j)}$ (by Lemma 1.1.1); thus $o(X_l) \geqslant m_i + m_j = o(X_i) + o(X_j)$ for all terms in the commutator. Now we may write

$$\begin{aligned} W_1 \dots W_k &= X^{\alpha_1} \dots W'[X_i, X_j]W'' \dots X^{\alpha_k} + X^{\alpha_1} \dots W' X_j X_i W'' \dots X^{\alpha_k} \\ &= \sum_l c_l X^{\alpha_1} \dots W' X_l W'' \dots X^{\alpha_k} + X^{\alpha_1} \dots W' X_j X_i W'' \dots X^{\alpha_k}. \end{aligned}$$

For each term in the sum we have $d < N$, so by induction its order is greater than or equal to

$$\begin{aligned} & o(X^{\alpha_1}) + \cdots + o(W') + o(X_l) + o(W'') + \cdots + o(X^{\alpha_k}) \\ &\quad = o(X^{\alpha_1}) + \cdots + (o(X^{\alpha_s}) - o(X_i)) + o(X_l) + (o(X^{\alpha_{s+1}}) - o(X_j)) + \cdots + o(X^{\alpha_k}) \\ &\quad \geqslant \sum_{i=1}^{k} o(W_i). \end{aligned}$$

Using (1A), we get

$$\begin{aligned} o(W_1 \dots W_k) &\geqslant \min\{o(\dots W' X_j X_i W'' \dots), o(\dots W' X_l W'' \dots)\} \\ &\geqslant \min\left\{o(\dots W' X_j X_i W'' \dots), \sum_{i=1}^{k} o(W_i)\right\}. \end{aligned}$$

Applying the transposition process repeatedly to the remaining term we preserve this inequality and can force this term to become a monomial, which must have the form X^β with $\beta_j = \Sigma_{i=1}^k \alpha_i(j)$, $1 \leqslant j \leqslant n$. Thus we get

$$o(W_1 \dots W_k) \geqslant \min\left\{o(X^\beta), \sum_{i=1}^{k} o(W_i)\right\}.$$

But $o(X^\beta) = \Sigma_{j=1}^n \beta_j o(X_j) = \Sigma_{i=1}^k o(W_i)$, as above, and this proves (1).

Let $\mathfrak{u}^m(\mathfrak{g}) = \{W \in \mathfrak{u}(\mathfrak{g}) : o(W) \geqslant m\}$. From (1), $\mathfrak{u}^m(\mathfrak{g})$ is an ideal of $\mathfrak{u}(\mathfrak{g})$. It has finite codimension, since $\mathfrak{u}^m(\mathfrak{g})$ contains every monomial having degree $\geqslant m$. Let $V = \mathfrak{u}(\mathfrak{g})/\mathfrak{u}^m(\mathfrak{g})$; choose a basis $\{W_1, \dots, W_r\}$ of V such that $W_1, \dots, W_{r_1}$ span $\mathfrak{u}^{m-1}(\mathfrak{g})/\mathfrak{u}^m(\mathfrak{g})$, $W_1, \dots, W_{r_2}$ span $\mathfrak{u}^{m-2}(\mathfrak{g})/\mathfrak{u}^m(\mathfrak{g})$, and so on. Then define $i: \mathfrak{g} \to \mathfrak{gl}(V)$ by

$$i(X)(W_j) = X W_j \pmod{\mathfrak{u}^m(\mathfrak{g})}.$$

Since $o(XW_j) \geqslant o(X) + o(W_j)$, we see that XW_j is a linear combination of $W_1, \dots, W_{j-1}$. Thus we have produced the subspaces of Engel's theorem besides showing that $i(X)$ is nilpotent. If $m > k$ then $i(X)I = X \neq 0$ for all $X \in \mathfrak{g}$, so that i is injective as well. □

Remark. The above proof gives a constructive (but unpleasant) procedure for writing a nilpotent Lie algebra as a subalgebra of some $\mathfrak{n}_n$; in fact, one can even compute a bound on n, in terms of k and dim $\mathfrak{g}$. In Examples 1.1.2 and 1.1.3, we gave explicit realizations for $\mathfrak{h}_n$, $\mathfrak{k}_n$.

We shall deal in Part II with a class of nilpotent Lie algebras called *graded*; in these, one can write $\mathfrak{g}=\mathfrak{g}_1\oplus\cdots\oplus\mathfrak{g}_r$, with $[\mathfrak{g}_i,\mathfrak{g}_j]\subseteq\mathfrak{g}_{i+j}$ (here $\mathfrak{g}_k=(0)$ if $k>r$). For these algebras, the properties of the function $o(X)$ are more transparent. The algebras $\mathbb{R}^n$, $\mathfrak{h}_n$, $\mathfrak{k}_n$, $\mathfrak{n}_n$, $\mathfrak{f}_{n,k}$ are all graded.

We turn now to two structural results about nilpotent Lie algebras. The first is simple, but is the key to much of the representation theory for nilpotent Lie groups. We shall repeatedly be giving proofs by induction on the dimension of the group G, or equivalently, on dim $\mathfrak{g}$. The proofs generally involve a dichotomy. If the center $\mathfrak{z}(\mathfrak{g})$ has dimension greater than 1, we can usually factor out a proper ideal $\mathfrak{h}\subseteq\mathfrak{z}(\mathfrak{g})$ and pass to the same problem on a quotient algebra of lower dimension. If dim $\mathfrak{z}(\mathfrak{g})=1$, we use the following lemma.

1.1.12 Lemma (Kirillov's Lemma). *Let $\mathfrak{g}$ be a noncommutative nilpotent Lie algebra whose center $\mathfrak{z}(\mathfrak{g})$ is one-dimensional. Then $\mathfrak{g}$ can be written as*

$$\mathfrak{g}=\mathbb{R}Z\oplus\mathbb{R}Y\oplus\mathbb{R}X\oplus\mathfrak{w}=\mathbb{R}X\oplus\mathfrak{g}_0,$$

a vector space direct sum, where

$$\mathbb{R}Z=\mathfrak{z}(\mathfrak{g}),\ \textit{and}\ [X,Y]=Z;$$

$$\mathfrak{g}_0=\mathbb{R}Y+\mathbb{R}Z+\mathfrak{w}\ \textit{is the centralizer of } Y\textit{, and an ideal.}$$

Proof. Since nilpotent Lie algebras of dimension $\leqslant 2$ are easily seen to be Abelian, we have dim $\mathfrak{g}\geqslant 3$. Let Z span $\mathfrak{z}(\mathfrak{g})$. Then $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})=\bar{\mathfrak{g}}$ is nilpotent and has dimension $\geqslant 2$. Pick a nonzero $\bar{Y}\in\mathfrak{z}(\bar{\mathfrak{g}})$, and let $Y\in\mathfrak{g}$ be a pre-image. For any $W\in\mathfrak{g}$, $[W,Y]\in\mathbb{R}Z$; thus we may define a linear map $\alpha:\mathfrak{g}\to\mathbb{R}$ by $[W,Y]=\alpha(W)Z$. The map α is not identically 0, since $Y\notin\mathfrak{z}(\mathfrak{g})$. Choose X with $\alpha(X)=1$, and let $\mathfrak{g}_0=\ker\alpha$. Then Y and Z are linearly independent elements of $\mathfrak{g}_0$. Let $\mathfrak{w}$ be a complementary subspace to $\mathbb{R}Y\oplus\mathbb{R}Z$ in $\mathfrak{g}_0$. The space $\mathfrak{g}_0$ is a subalgebra because if $W_1, W_2\in\mathfrak{g}_0$, we have

$$[[W_1,W_2],Y]=-[W_2,[W_1,Y]]+[W_1,[W_2,Y]]=0,$$

so

$$[W_1,W_2]\in\mathfrak{g}_0.$$

Lemma 1.1.8 implies that $\mathfrak{g}_0$ is an ideal. □

The final result of this section concerns 'nice' bases for nilpotent Lie algebras. We shall use these bases repeatedly, both for calculations and for various aspects of the theory. Their relation to coordinates on nilpotent Lie groups will take up a large part of the next section.

1.1.13 Theorem. *Let $\mathfrak{g}$ be a nilpotent Lie algebra, and let $\mathfrak{g}_1 \subseteq \mathfrak{g}_2 \subseteq \cdots \subseteq \mathfrak{g}_k \subseteq \mathfrak{g}$ be subalgebras, with* $\dim \mathfrak{g}_j = m_j$ *and* $\dim \mathfrak{g} = n$.

(a) *$\mathfrak{g}$ has a basis $\{X_1, \ldots, X_n\}$ such that*
  (i) *for each m, $\mathfrak{h}_m = \mathbb{R}\text{-span}\{X_1, \ldots, X_m\}$ is a subalgebra of $\mathfrak{g}$,*
  (ii) *for $1 \leqslant j \leqslant k$, $\mathfrak{h}_{m_j} = \mathfrak{g}_j$.*
(b) *If the $\mathfrak{g}_j$ are ideals of $\mathfrak{g}$, then one can pick the X_j so that (i) is replaced by*
  (iii) *for each m, $\mathfrak{h}_m = \mathbb{R}\text{-span}\{X_1, \ldots, X_m\}$ is an ideal of $\mathfrak{g}$.*

Proof. For subalgebras, it suffices to show that if $\mathfrak{h}$ is a proper subalgebra of $\mathfrak{g}$, then there is an element $X \in \mathfrak{g}$ such that $\mathfrak{h} \oplus \mathbb{R}X$ is a subalgebra of $\mathfrak{g}$.

One way to find X is the following: define the normalizer of $\mathfrak{h}$ to be $\mathfrak{n}(\mathfrak{h}) = \{X \in \mathfrak{g} : [X, \mathfrak{h}] \subseteq \mathfrak{h}\}$. Then $\mathfrak{h}$ is properly contained in $\mathfrak{n}(\mathfrak{h})$ if $\mathfrak{h} \neq \mathfrak{g}$. For there is some m such that $g^{(m)}$ is not contained within $\mathfrak{h}$. Choose m to be maximal with this property. Then $[\mathfrak{g}^{(m)}, \mathfrak{h}] \subseteq [\mathfrak{g}^{(m)}, \mathfrak{g}] = \mathfrak{g}^{(m+1)} \subseteq \mathfrak{h}$, so that $\mathfrak{g}^{(m)} \subseteq \mathfrak{n}(\mathfrak{h})$ and $\mathfrak{g}^{(m)} \sim \mathfrak{h} \neq \emptyset$. Now choose $X \in \mathfrak{n}(\mathfrak{h}) \sim \mathfrak{h}$, and part (a) is done.

For ideals, it suffices to show that $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_1 \neq (0)$, take quotients by the central ideal, and use induction on $\dim \mathfrak{g}$. So let $\mathfrak{g}_1^1 = \mathfrak{g}_1, \mathfrak{g}_1^{j+1} = [\mathfrak{g}, \mathfrak{g}_1^j]$. Since $\mathfrak{g}_1^j \subseteq \mathfrak{g}^{(j)}$, there is a smallest r with $\mathfrak{g}_1^{r+1} = (0)$. Then $(0) \neq \mathfrak{g}_1^r \subseteq \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_1$. □

Note. Of course, the $\mathfrak{h}_m$ in this theorem are not necessarily Heisenberg algebras.

Note. We call a basis satisfying (i) and (ii) a *weak Malcev basis* for $\mathfrak{g}$ through $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$, and one satisfying (ii) and (iii), a *strong Malcev basis* for $\mathfrak{g}$ through $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$. We shall often use this theorem when $k = 0$, in which case we simply refer to a *weak* (*or strong*) *Malcev basis* for $\mathfrak{g}$.

We list the main auxiliary results in the above proof as corollaries.

1.1.14 Corollary. *If $\mathfrak{h}$ is a proper subalgebra of a nilpotent Lie algebra $\mathfrak{g}$, then the normalizer of $\mathfrak{h}$ strictly contains $\mathfrak{h}$.*

1.1.15 Corollary. *If $\mathfrak{h}$ is a nonzero ideal of a nilpotent Lie algebra $\mathfrak{g}$, then* $\mathfrak{h} \cap \mathfrak{z}(\mathfrak{g}) \neq (0)$.

Remark. Corollary 1.1.14 provides an alternative proof of Lemma 1.1.8.

1.2 Nilpotent Lie groups

A nilpotent Lie group G is one whose Lie algebra $\mathfrak{g}$ is nilpotent. In this work we shall always assume that G is connected and usually that it is simply