1
Introduction

1.1. The classical Fredholm theory

For convenience, as in the theory of normed spaces, we shall use the word 'scalar' to mean either 'real number' or 'complex number' according to taste. But, of course, this terminology must be used consistently, the same meaning being given to the word 'scalar' throughout any connected piece of argument.

The classical Fredholm theory (see, for instance, Fredholm, 1899, 1903; for other references see §8.1) deals with the solution of an equation of the type

$$x(s) = y(s) + \lambda \int_a^b k(s,t)x(t)\,dt \quad (a \leqslant s \leqslant b),$$

where k is a given continuous scalar function of two real variables with domain $[a,b] \times [a,b]$ (for our use of the word 'function' see §8.1), y is a given continuous scalar function of a single real variable with domain $[a,b]$, λ is a scalar, and the continuous scalar function x of a single real variable with domain $[a,b]$ is to be determined (in much of the literature the scalars are taken to be real, but the extension to complex scalars is straightforward). Such an equation is known as a **Fredholm integral equation** ('of the second kind'). We call the function k the **Fredholm-kernel** of the equation (and of the associated operator K given by the equation

$$(Kx)(s) = \int_a^b k(s,t)x(t)dt \quad (a \leqslant s \leqslant b)$$

for every continuous scalar function x with domain $[a, b]$); many writers call it the 'kernel' of the equation (and of the operator), but we shall be using this word in the algebraic sense (cf., for instance, Greub, 1967a, p. 41).

For convenience, we denote by

$$k\begin{pmatrix} s_1, s_2, \ldots, s_n \\ t_1, t_2, \ldots, t_n \end{pmatrix}$$

the determinant

$$\begin{vmatrix} k(s_1,t_1) & k(s_1,t_2) & \cdots & k(s_1,t_n) \\ k(s_2,t_1) & k(s_2,t_2) & \cdots & k(s_2,t_n) \\ \cdots & \cdots & \cdots & \cdots \\ k(s_n,t_1) & k(s_n,t_2) & \cdots & k(s_n,t_n) \end{vmatrix}$$

for $n=0,1,2,\ldots$, and for $a \leqslant s_1, s_2, \ldots, s_n, t_1, t_2, \ldots, t_n \leqslant b$ (for $n=0$, we interpret this, the sum of 0! terms, each the product of 0 factors, conventionally as 1, cf. the remark after Definition 3.1.3). We observe that k, being a continuous function with compact domain, is bounded. Let M be a bound for k, that is $|k(s,t)| \leqslant M$ when $a \leqslant s, t \leqslant b$. Then, by Hadamard's determinant theorem (cf. §8.1), we have

$$\left| k\begin{pmatrix} s_1, s_2, \ldots, s_n \\ t_1, t_2, \ldots, t_n \end{pmatrix} \right| \leqslant n^{n/2} M^n$$

when $a \leqslant s_1, s_2, \ldots, s_n, t_1, t_2, \ldots, t_n \leqslant b$, for $n = 0, 1, 2, \ldots$.

We now consider the power series

$$k\begin{pmatrix} s_1, \ldots, s_n \\ t_1, \ldots, t_n \end{pmatrix} - \lambda \int_a^b k\begin{pmatrix} s_1, \ldots, s_n, u_1 \\ t_1, \ldots, t_n, u_1 \end{pmatrix} du_1$$
$$+ \frac{\lambda^2}{2!} \int_a^b \int_a^b k\begin{pmatrix} s_1, \ldots, s_n, u_1, u_2 \\ t_1, \ldots, t_n, u_1, u_2 \end{pmatrix} du_1 du_2 - \cdots$$
$$= \sum_{\nu=0}^{\infty} \lambda^\nu \frac{(-1)^\nu}{\nu!} \int_a^b \cdots \int_a^b k\begin{pmatrix} s_1, \ldots, s_n, u_1, \ldots, u_\nu \\ t_1, \ldots, t_n, u_1, \ldots, u_\nu \end{pmatrix} du_1 \ldots du_\nu$$

for all points $s_1, \ldots, s_n, t_1, \ldots, t_n$ of $[a,b]$ and for every scalar λ, for $n = 0, 1, 2, \ldots$. From the above inequality it follows that the modulus of the coefficient of λ^ν in this series is at most

$$\frac{1}{\nu!}(n+\nu)^{(n+\nu)/2} M^{n+\nu} (b-a)^\nu$$

for all points $s_1, \ldots, s_n, t_1, \ldots, t_n$ of $[a,b]$, for $n, \nu = 0, 1, 2, \ldots$, and so, by the Cauchy–Hadamard theorem (cf. §8.1), the power series has infinite radius of convergence for $n = 0, 1, 2, \ldots$. We denote its sum by

$$d\left(\begin{matrix} s_1, \ldots, s_n \\ t_1, \ldots, t_n \end{matrix} \middle| \lambda \right)$$

for all points $s_1, \ldots, s_n, t_1, \ldots, t_n$ of $[a,b]$ and for every scalar λ, for $n = 0, 1, 2, \ldots$. It is customary to write $d(\lambda)$ when $n = 0$ and $d(s, t; \lambda)$ when $n = 1$. We call it the **Fredholm determinant** when $n = 0$ and the nth **Fredholm minor** when $n > 0$.

Plemelj (1904, p. 122) has given formulae for the coefficients d_ν and $d_\nu(s,t)$ of

λ^ν $(\nu = 0, 1, 2, \ldots)$ in the series for $d(\lambda)$ and $d(s, t; \lambda)$ respectively in terms of the iterates $k^{(n)}$ of the Fredholm-kernel k given by the equation

$$k^{(n)}(s,t) = \int_a^b \int_a^b \cdots \int_a^b \int_a^b k(s,u_1)k(u_1,u_2)\ldots k(u_{n-2},u_{n-1})k(u_{n-1},t)\,du_1\,du_2\ldots du_{n-2}\,du_{n-1}$$

for $a \leqslant s, t \leqslant b$, for $n = 1, 2, 3, \ldots$. The Plemelj formulae are

$$d_\nu = \frac{(-1)^\nu}{\nu!} \begin{vmatrix} \sigma_1 & \nu-1 & 0 & \cdots & 0 & 0 \\ \sigma_2 & \sigma_1 & \nu-2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma_{\nu-2} & \sigma_{\nu-3} & \sigma_{\nu-4} & \cdots & 2 & 0 \\ \sigma_{\nu-1} & \sigma_{\nu-2} & \sigma_{\nu-3} & \cdots & \sigma_1 & 1 \\ \sigma_\nu & \sigma_{\nu-1} & \sigma_{\nu-2} & \cdots & \sigma_2 & \sigma_1 \end{vmatrix}$$

and

$$d_\nu(s,t) = \frac{(-1)^\nu}{\nu!} \begin{vmatrix} k(s,t) & \nu & 0 & 0 & \cdots & 0 & 0 \\ k^{(2)}(s,t) & \sigma_1 & \nu-1 & 0 & \cdots & 0 & 0 \\ k^{(3)}(s,t) & \sigma_2 & \sigma_1 & \nu-2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ k^{(\nu-1)}(s,t) & \sigma_{\nu-2} & \sigma_{\nu-3} & \sigma_{\nu-4} & \cdots & 2 & 0 \\ k^{(\nu)}(s,t) & \sigma_{\nu-1} & \sigma_{\nu-2} & \sigma_{\nu-3} & \cdots & \sigma_1 & 1 \\ k^{(\nu+1)}(s,t) & \sigma_\nu & \sigma_{\nu-1} & \sigma_{\nu-2} & \cdots & \sigma_2 & \sigma_1 \end{vmatrix},$$

where

$$\sigma_r = \int_a^b k^{(r)}(u,u)\,du \quad (r = 1, 2, 3, \ldots),$$

for $\nu = 0, 1, 2, \ldots$, and for $a \leqslant s, t \leqslant b$. Analogues of the Plemelj formulae play an important part in Chapters 4 and 7 (cf. Definition 4.3.1).

The Fredholm solution of the Fredholm integral equation for a given value of the (scalar) parameter λ is as follows. There is a least non-negative integer d, called the **deficiency** of λ, such that the function

$$d\left(\begin{matrix} s_1, \ldots, s_d \\ t_1, \ldots, t_d \end{matrix} \middle| \lambda\right)$$

is not identically zero in the points $s_1, \ldots, s_d$, $t_1, \ldots, t_d$ of $[a, b]$. Choose, then, fixed points $s_1, \ldots, s_d$, $t_1, \ldots, t_d$ from $[a, b]$ for which the above function does not vanish. Then the functions $\xi_1, \ldots, \xi_d$ given by the equations

$$\xi_1(s) = d\left(\begin{matrix} s, s_2, \ldots, s_d \\ t_1, t_2, \ldots, t_d \end{matrix} \middle| \lambda\right), \ldots, \xi_d(s) = d\left(\begin{matrix} s_1, \ldots, s_{d-1}, s \\ t_1, \ldots, t_{d-1}, t_d \end{matrix} \middle| \lambda\right)$$

for all points s of $[a, b]$ form a base for the set of solutions x of the homogeneous equation

$$x(s) = \lambda \int_a^b k(s,t)x(t)\,dt \quad (a \leqslant s \leqslant b),$$

and similarly the functions $\phi_1, \ldots, \phi_d$ given by the equations

$$\phi_1(t) = d\left(\begin{matrix} s_1, s_2, \ldots, s_d \\ t, t_2, \ldots, t_d \end{matrix}\middle|\lambda\right), \ldots, \phi_d(t) = d\left(\begin{matrix} s_1, \ldots, s_{d-1}, s_d \\ t_1, \ldots, t_{d-1}, t \end{matrix}\middle|\lambda\right)$$

for all points t of $[a, b]$ form a base for the set of solutions f of the conjugate homogeneous equation

$$f(t) = \lambda \int_a^b f(s)k(s,t)\,ds \quad (a \leqslant t \leqslant b).$$

The equation

$$x(s) = y(s) + \lambda \int_a^b k(s,t)x(t)\,dt \quad (a \leqslant s \leqslant b)$$

is then soluble if and only if

$$\int_a^b f(u)y(u)\,du = 0$$

for every solution f of the equation

$$f(t) = \lambda \int_a^b f(s)k(s,t)\,ds \quad (a \leqslant t \leqslant b),$$

that is if and only if

$$\int_a^b \phi_i(u)y(u)\,du = 0$$

for $i = 1, \ldots, d$, and the general solution is then given by the equation

$$x(s) = y(s) + \lambda \int_a^b \frac{d\left(\begin{matrix} s, s_1, \ldots, s_d \\ t, t_1, \ldots, t_d \end{matrix}\middle|\lambda\right)}{d\left(\begin{matrix} s_1, \ldots, s_d \\ t_1, \ldots, t_d \end{matrix}\middle|\lambda\right)} y(t)dt + \sum_{i=1}^{d} c_i \xi_i(s)$$

for $a \leqslant s \leqslant b$, where $c_1, \ldots, c_d$ are arbitrary constants.

In establishing these conclusions, we use various properties of the functions introduced above. For instance, we have, for each scalar λ, and for $n = 1, 2, 3, \ldots$, the relations

$$d\left(\begin{matrix} s_1, s_2, \ldots, s_n \\ t_1, t_2, \ldots, t_n \end{matrix}\middle|\lambda\right) - \lambda\int_a^b k(s_1, u)\, d\left(\begin{matrix} u, s_2, \ldots, s_n \\ t_1, t_2, \ldots, t_n \end{matrix}\middle|\lambda\right) du$$

$$= \sum_{r=1}^{n} (-1)^{r-1} k(s_1, t_r)\, d\left(\begin{matrix} s_2, \ldots, s_r, s_{r+1}, \ldots, s_n \\ t_1, \ldots, t_{r-1}, t_{r+1}, \ldots, t_n \end{matrix}\middle|\lambda\right)$$

and

$$d\left(\begin{matrix} s_1, s_2, \ldots, s_n \\ t_1, t_2, \ldots, t_n \end{matrix}\middle|\lambda\right) - \lambda\int_a^b d\left(\begin{matrix} s_1, s_2, \ldots, s_n \\ u, t_2, \ldots, t_n \end{matrix}\middle|\lambda\right) k(u, t_1)\, du$$

$$= \sum_{r=1}^{n} (-1)^{r-1} k(s_r, t_1)\, d\left(\begin{matrix} s_1, \ldots, s_{r-1}, s_{r+1}, \ldots, s_n \\ t_2, \ldots, t_r, t_{r+1}, \ldots, t_n \end{matrix}\middle|\lambda\right)$$

for all points $s_1, s_2, \ldots, s_n, t_1, t_2, \ldots, t_n$ of $[a, b]$. These are related to Lovitt's 'Fredholm's fundamental relations' (1924, pp. 27–8, 34–8, 49–50). It was these relations that suggested the operations '$\wedge$' and '$\vee$' on tensors introduced in §4.1 (cf. remark before Definition 4.1.5, and §8.4). Kowalewski (1930, pp. 195–6) also gave further relations which suggested relations that proved valuable in particular in §§4.4 and 6.4, namely, for each scalar λ, and for $n = 1, 2, 3, \ldots$,

$$d(\lambda)\, d\left(\begin{matrix} s_1, s_2, \ldots, s_n \\ t_1, t_2, \ldots, t_n \end{matrix}\middle|\lambda\right)$$

$$= \sum_{r=1}^{n} (-1)^{r-1} d(s_1, t_r; \lambda)\, d\left(\begin{matrix} s_2, \ldots, s_r, s_{r+1}, \ldots, s_n \\ t_1, \ldots, t_{r-1}, t_{r+1}, \ldots, t_n \end{matrix}\middle|\lambda\right)$$

$$= \sum_{r=1}^{n} (-1)^{r-1} d(s_r, t_1; \lambda)\, d\left(\begin{matrix} s_1, \ldots, s_{r-1}, s_{r+1}, \ldots, s_n \\ t_2, \ldots, t_r, t_{r+1}, \ldots, t_n \end{matrix}\middle|\lambda\right)$$

for all points $s_1, s_2, \ldots, s_n, t_1, t_2, \ldots, t_n$ of $[a, b]$.

1.2. Summary

In the first, introductory, chapter, we outline the classical Fredholm theory, which is the starting point of the theory of this tract. Then, after this summary, we make a preliminary study of the complexification of a real normed space. This will be used in later chapters (especially Chapter 5) to enable us to apply to real Banach spaces results which are essentially complex analytic in their nature. Among the topics considered here are admissible norms, conjugate spaces and operators. Further developments are mentioned where they are relevant.

In the body of this tract, instead of continuous scalar functions x and y, we consider points x and y of a Banach space $\mathfrak{X}$. The Fredholm integral equation considered in §1.1 is then replaced by the Fredholm equation

$$x = y + \lambda Kx,$$

where the operator K on $\mathfrak{X}$ is given, the point y of $\mathfrak{X}$ is given, the scalar λ is given, and the point x of $\mathfrak{X}$ is to be determined. Here, as elsewhere, in the absence of any statement to the contrary, $\mathfrak{X}$ may be either a real Banach space or a complex Banach space, with the obvious conventions.

The main aim of the tract is to find analogues of the Fredholm determinant $d(.)$ and the Fredholm minors

$$d\left(\begin{matrix} s_1, s_2, \ldots, s_n \\ t_1, t_2, \ldots, t_n \end{matrix}\middle| .\right)$$

for $n = 1, 2, 3, \ldots$ in terms of which the above equation can be solved. We also use these and related formulae to study the Riesz theory. Among other things, we set up bases for the null spaces that appear in that theory. Earlier work (prior to my starting work on the subject in 1947) was due to Michal & Martin (1934) and Frank Smithies (1941), though, in both cases, only those parts of the theory which concern the Fredholm determinant and the first Fredholm minor were considered (cf. also Ruston, 1951a).

We now describe in a little more detail the subject-matter of the remaining chapters.

In Chapter 2 we consider asymptotically quasi-compact (aqc) operators on a normed space. Various structural questions are considered, but much of the chapter is devoted to the Riesz theory, for which the space concerned is assumed to be complete. In this connection, we consider Riesz points of an operator and Riesz operators. It is shown that an operator is a Riesz operator if and only if it is aqc.

In Chapter 3 we consider the theory of tensor products of normed spaces (we do not restrict consideration, as we did in Ruston, 1951b, to tensor products of Banach spaces, since a tensor product of Banach spaces, when furnished with a particular norm, is not in general complete – the case in which all the factors are finite-dimensional is an obvious exception). The theory goes back to work of Schatten (see references in §8.3) which dealt exclusively with tensor products of two factors. The development here is carried further than is strictly necessary for the later chapters in order to give a more complete picture of the theory.

In Chapter 4 we introduce the notion of an (m, n)-tensor on a Banach space $\mathfrak{X}$, where m and n are non-negative integers. An (m, n)-tensor on $\mathfrak{X}$ is a bounded linear mapping of a certain completed tensor nth power of $\mathfrak{X}$ into a certain completed tensor mth power of $\mathfrak{X}$. Certain operations are defined for these, suggested by relations which play an important part in the classical Fredholm theory. Attention then turns to operators of finite rank on a Banach space $\mathfrak{X}$. We define

(n, n)-tensors (n-operators) $D_n(\lambda)$ analogous to

$$d\left(\begin{matrix} s_1, s_2, \ldots, s_n \\ t_1, t_2, \ldots, t_n \end{matrix}\middle|\, \lambda\right)$$

in the classical theory. Then a Fredholm theory for operators of finite rank is developed, in which $D_n(\lambda)$ plays a role analogous to that played by the above-mentioned analogue in the classical theory.

In Chapter 5 we introduce the notion of a Fredholm determinant for an operator on a Banach space $\mathfrak{X}$, that is a scalar integral function, not identically zero, such that certain power series, defined in terms of this function and the operator, have infinite radius of convergence. These power series, and other related ones, define certain tensor-valued integral functions, which we call Fredholm formulae (sometimes called 'formulae of Fredholm type', cf. Ruston, 1953b). Then there is a Fredholm theory for operators on $\mathfrak{X}$ with a Fredholm determinant in terms of these Fredholm formulae, which bears some resemblance to the theory for operators of finite rank referred to above. It is shown that an operator has a Fredholm determinant if and only if it is a Riesz operator, and so if and only if it is aqc.

In Chapter 6 we study relations between the Fredholm formulae introduced in the last chapter and the Riesz theory. In particular, we present a number of bases for the null spaces that feature in the Riesz theory.

In Chapter 7 we present a number of theories which give methods of constructing Fredholm formulae for certain classes of operators on a general Banach space. We consider the theory of the trace class of nuclear operators, and also the theories associated with the names of Leżański and Sikorski, of Grothendieck, and of Saphar, and combinations of these theories. These theories have been freely adapted to fit in with the spirit of this tract.

The last chapter, Chapter 8, consists of notes and comments on the other chapters, additional to those in the body of the text, and also some remarks on possible further developments. Each section of this chapter, except the last, corresponds to one of the other chapters, and bears the same number and title. The notes and comments are collected here to avoid breaking up the free flow of the text by innumerable footnotes and references. Notes and comments have only been included in the body of the text when, in my opinion, they had a positive contribution to make to the understanding of the context.

1.3. The complexification of real normed spaces

I shall be more formal in this section than in the earlier sections.

We first consider the complexification of a real vector space.

Theorem 1.3.1. *Let $\mathfrak{X}$ be a real vector space. Let $\mathfrak{X}_C$ be the set of all ordered pairs $z=(x,y)$ of points of $\mathfrak{X}$ together with the operations given by the equations*

$$(x,y)+(u,v)=(x+u,y+v),$$

and

$$(\alpha+i\beta)(x,y)=(\alpha x-\beta y,\alpha y+\beta x),$$

for all points x, y, u, v of $\mathfrak{X}$ and all real numbers α and β. Then $\mathfrak{X}_C$ is a complex vector space.

The proof of this theorem is a routine matter.

Definition 1.3.1. *Let $\mathfrak{X}$ be a real vector space. Then the complex vector space $\mathfrak{X}_C$ constructed in* Theorem 1.3.1 *is called the* ***complexification*** *of $\mathfrak{X}$.*

Notice that $(x,y)=(x,0)+i(y,0)$ for all points x, y of $\mathfrak{X}$. It is sometimes convenient to 'identify' each point x of $\mathfrak{X}$ with the point $(x,0)$ of $\mathfrak{X}_C$, thus 'embedding' $\mathfrak{X}$ in $\mathfrak{X}_C$. We then have $(x,y)=x+iy$ for all points x, y of $\mathfrak{X}$.

Theorem 1.3.2. *Let $\mathfrak{A}$ be a real algebra. Let $\mathfrak{A}_C$ be the complexification of $\mathfrak{A}$ qua vector space (as in* Definition 1.3.1*) together with the operation given by the equation*

$$(x,y)(u,v)=(xu-yv,xv+yu),$$

for all points x, y, u, v of $\mathfrak{A}$. Then $\mathfrak{A}_C$ is a complex algebra. Moreover, (e,f) (where $e,f\in\mathfrak{A}$) is a unit for $\mathfrak{A}_C$ if and only if e is a unit for $\mathfrak{A}$ and $f=0$.

(The word 'qua' means in effect 'regarded as'.)

Just as we call an additive identity a **zero**, so we call a multiplicative identity a **unit** (cf. Bonsall & Duncan, 1973b, p. 10).

The proof of the theorem is straightforward (cf. Rickart, 1960, p. 6; Bonsall & Duncan, 1973b, p. 68).

Definition 1.3.2. *Let $\mathfrak{A}$ be a real algebra. Then the complex algebra $\mathfrak{A}_C$ constructed in* Theorem 1.3.2 *is called the* ***complexification*** *of $\mathfrak{A}$.*

Let $\mathfrak{X}$ be a real vector space, and let $\mathfrak{M}$ be a subspace of $\mathfrak{X}$. Then $\mathfrak{M}_C$ is a subspace of $\mathfrak{X}_C$.

Similarly, if $\mathfrak{A}$ is a real algebra, $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$, and $\mathfrak{I}$ is an ideal in $\mathfrak{A}$, then $\mathfrak{B}_C$ is a subalgebra of $\mathfrak{A}_C$, and $\mathfrak{I}_C$ is an ideal in $\mathfrak{A}_C$.

Theorem 1.3.3. *Let $\mathfrak{X}$ be a real vector space, and let $\mathfrak{M}$ and $\mathfrak{N}$ be subspaces of $\mathfrak{X}$ with $\mathfrak{M} \subseteq \mathfrak{N}$.*

(i) *Let $(x_\alpha)_{\alpha\in A}$ be a family of points of $\mathfrak{X}$. Then $(x_\alpha)_{\alpha\in A}$ is linearly independent modulo $\mathfrak{M}$ if and only if $((x_\alpha, 0))_{\alpha\in A}$ is linearly independent modulo $\mathfrak{M}_C$.*

(ii) *Let $(x_\alpha)_{\alpha\in A}$ be a family of points of $\mathfrak{N}$. Then $(x_\alpha)_{\alpha\in A}$ spans $\mathfrak{N}$ modulo $\mathfrak{M}$ if and only if $((x_\alpha, 0))_{\alpha\in A}$ spans $\mathfrak{N}_C$ modulo $\mathfrak{M}_C$.*

(iii) *Let $(x_\alpha)_{\alpha\in A}$ be a family of points of $\mathfrak{N}$. Then $(x_\alpha)_{\alpha\in A}$ is a base for $\mathfrak{N}$ modulo $\mathfrak{M}$ if and only if $((x_\alpha, 0))_{\alpha\in A}$ is a base for $\mathfrak{N}_C$ modulo $\mathfrak{M}_C$.*

For definitions of the concepts occurring above, see §8.6
The proof is straightforward.

Corollary. *Let $\mathfrak{X}$ be a real vector space. Then $\mathfrak{X}$ is finite-dimensional if and only if $\mathfrak{X}_C$ is finite-dimensional, and then the number of (real) dimensions of $\mathfrak{X}$ is equal to the number of (complex) dimensions of $\mathfrak{X}_C$.*

The proof is, again, straightforward.

We now turn our attention to real normed spaces.

When $\mathfrak{X}$ is a real normed space, we shall call $\mathfrak{X}_C$ (as defined in Definition 1.3.1) the **vector space complexification** of $\mathfrak{X}$. Similarly, when $\mathfrak{A}$ is a real normed algebra, we shall call $\mathfrak{A}_C$ (as defined in Definition 1.3.2) the **algebra complexification** of $\mathfrak{A}$.

Note. In discussing various norms on $\mathfrak{X}_C$, it is sometimes convenient to use the well-known fact that, if a and b are real numbers with $a^2 + b^2 = 1$, then there is a real number α (unique, in fact, modulo multiples of 2π) with $\cos\alpha = a$ and $\sin\alpha = b$ (such a number α is given by the equations $\alpha = \int_0^{b/a}(1+t^2)^{-1}\,dt$ when $a > 0$, $\alpha = \frac{1}{2}\pi \operatorname{sgn} b$ when $a = 0$, and $\alpha = \pi \operatorname{sgn} b + \int_0^{b/a}(1+t^2)^{-1}\,dt$ when $a < 0$ – here, as elsewhere, I denote by $\operatorname{sgn} z$ the **signum** of the complex number z, defined to be zero when z is zero, and to be $z/|z|$ when z is non-zero).

The norms on $\mathfrak{X}_C$ that we shall be interested in are those satisfying the following definition.

Definition 1.3.3. *Let $\mathfrak{X}$ be a real normed space, let $\mathfrak{X}_C$ be the vector space complexification of $\mathfrak{X}$, and let $\|\,.\,\|$ be a norm on $\mathfrak{X}_C$. Then $\|\,.\,\|$ is said to be* ***admissible*** *iff*

$$\max(\|x\|, \|y\|) \leqslant \|(x, y)\| \leqslant \|x\| + \|y\|$$

for all points x, y of $\mathfrak{X}$. The vector space $\mathfrak{X}_C$ together with this norm on it is called ***a complexification*** *of the real normed space $\mathfrak{X}$.*

Following Halmos, I use 'iff' in a formal or informal definition when 'if' is grammatically correct but the logical meaning is 'if and only if'.

There is not, in general, a unique admissible norm of $\mathfrak{X}_C$, but note that, since

$$\max(\|x\|, \|y\|) \leqslant \|x\| + \|y\| \leqslant 2\max(\|x\|, \|y\|)$$

for all points x, y of $\mathfrak{X}$, all admissible norms on $\mathfrak{X}_C$ are equivalent (Brown & Page, 1970, p. 110).

Theorem 1.3.4. *Let $\mathfrak{X}$ be a real normed space, let $\mathfrak{X}_C$ be the vector space complexification of $\mathfrak{X}$, and let $\|.\|$ be a norm on $\mathfrak{X}_C$ such that*

$$\max(\|x\|, \|y\|) \leqslant \|(x, y)\|$$

for all points x, y of $\mathfrak{X}$. Then $\|.\|$ is admissible if and only if

$$\|(x, 0)\| = \|x\|$$

for every point x of $\mathfrak{X}$.

The proof is immediate.

Theorem 1.3.5. *Let $\mathfrak{X}$ be a real normed space, and let $\mathfrak{X}_C$ be the vector space complexification of $\mathfrak{X}$. Then:*

(i) *the equation*

$$\|(x, y)\| = \sup_{-\pi \leqslant \theta \leqslant \pi} \|x\cos\theta + y\sin\theta\|$$

for all points x, y of $\mathfrak{X}$ defines the least admissible norm $\|.\|$ on $\mathfrak{X}_C$;

(ii) *the equation*

$$\|(x, y)\| = \inf \sum_{r=1}^{n} |\alpha_r + i\beta_r| \, \|x_r\|$$

for all points x, y of $\mathfrak{X}$, the infimum being taken over all finite families of triples (α_r, β_r, x_r) $(r = 1, 2, \ldots, n)$ with α_r and β_r real numbers and x_r a point of $\mathfrak{X}$ for each r and with $\sum_{r=1}^{n} \alpha_r x_r = x$ and $\sum_{r=1}^{n} \beta_r x_r = y$ (such families exist, e.g. $((1, 0, x), (0, 1, y))$), defines the greatest admissible norm $\|.\|$ on $\mathfrak{X}_C$.

Proof. (i) The proof that $\|.\|$ is an admissible norm on $\mathfrak{X}_C$ is straightforward. That it is less than or equal to any admissible norm $\|.\|'$ on $\mathfrak{X}_C$ follows from the fact that

$$\|x\cos\theta + y\sin\theta\| \leqslant \|(\cos\theta - i\sin\theta)(x, y)\|' = \|(x, y)\|'$$

for all points x, y of $\mathfrak{X}$ and every real number θ.