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Eiichi Abe
Excerpt
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Modules and algebras

1 Modules

This section deals with direct sums, direct products, tensor products, and the projective and inductive limits of modules. Proofs of some fundamental properties of such constructions have been omitted and left as exercises. For these, the reader is asked to refer to texts such as [3] or [5].

1.1 Modules

A set A with two operations – addition and multiplication – which satisfies properties (1) to (3) below is said to be a **ring with identity**. Since this book deals exclusively with this type of ring, we will call them simply **rings**.

(1) The addition $+$ makes A an abelian group.
(2) The multiplication $\cdot$ makes A a semigroup with identity element 1.
(3) The distributive law holds. Namely, for $a, b, c \in A$, we have

$$a\cdot(b+c)=a\cdot b+a\cdot c, \quad (a+b)\cdot c=a\cdot c+b\cdot c.$$

A ring with a commutative multiplication is called a **commutative ring**. Henceforth, the product $a\cdot b$ of $a, b\in A$ will be written ab. Let A and B be rings. If a map $u: A\rightarrow B$ satisfies the properties

$$u(a+b)=u(a)+u(b), \quad u(ab)=u(a)u(b), \quad u(1)=1, \quad a, b\in A,$$

then u is said to be a **ring morphism** from A to B. The category of rings (resp. commutative rings) will be denoted **Alg** (resp. **M**) and the set of all ring morphisms from A to B will be written **Alg** (A, B) (resp. $\mathbf{M}(A, B)$ when A, B are commutative rings).

For a ring A and an abelian group M, suppose we are given a map $\varphi: A\times M\rightarrow M$ (resp. $\psi: M\times A\rightarrow M$). We signify the group operation on M by addition $+$, and for $a \in A$, $x \in M$, we write $\varphi(a, x) = ax$ (resp. $\psi(x, a) = xa$). When conditions (1) to (4) (resp. (1′) to (4′)) below hold for $a, b \in A$ and $x, y \in M$, M is called a **left A-module** (resp. **right A-module**), and φ (resp. ψ) is said to be the **structure map** of the left (resp. right) A-module.

(1) $a(x + y) = ax + ay$, resp. (1′) $(x + y)a = xa + ya$,
(2) $(a + b)x = ax + bx$, (2′) $x(a + b) = xa + xb$,
(3) $(ab)x = a(bx)$, (3′) $x(ab) = (xa)b$,
(4) $1x = x$. (4′) $x1 = x$.

Moreover, given rings A, B, if M is both a left A-module and a right B-module satisfying the condition

$$(ax)b = a(xb), \quad a \in A, \quad b \in B, \quad x \in M,$$

then M is called a **two-sided (A, B)-module**. A two-sided (A, A)-module is called simply a **two-sided A-module**. If A is a commutative ring, a left A-module can be regarded as a right A-module, and is often simply called an A-module. For instance, an abelian group is a $\mathbb{Z}$-module. In the case of a ring A, when the map defining the multiplication $\mu : A \times A \to A$ given by $\mu(a, b) = ab$, $a, b \in A$ is taken to be the structure map, A becomes a left A-module as well as a right A-module. Moreover, A is a two-sided A-module. For a field k, a k-module is also called a **k-linear space** or a **k-vector space**.

Let M, N be left A-modules. A map $f : M \to N$ such that

$$f(ax + by) = af(x) + bf(y), \quad a, b \in A, \quad x, y \in M,$$

is called a **left A-module morphism** from M to N. If k is a field, a k-module morphism is sometimes called a **k-linear map**. The category of left A-modules is denoted ${}_A\mathbf{Mod}$, and the set of all left A-module morphisms from M to N is denoted ${}_A\mathbf{Mod}(M, N)$. Similarly, given right A-modules M, N, we can define **right A-module morphisms** and the set of all right A-module morphisms from M to N, which we write $\mathbf{Mod}_A(M, N)$. In particular, $\mathbf{Mod}_A(M, M)$ is written $\mathrm{End}_A(M)$. If $f \in {}_A\mathbf{Mod}(M, N)$ or $f \in \mathbf{Mod}_A(M, N)$ is bijective, f is said to be an **isomorphism**. The identity map from M to M is an isomorphism, denoted by 1_M or simply by 1.

Now let $f, g \in {}_A\mathbf{Mod}(M, N)$. Defining

$$(f+g)(x) = f(x) \pm g(x), \quad x \in M,$$

$f \pm g$ becomes a left A-module morphism from M to N. Under this operation, ${}_A\mathbf{Mod}(M, N)$ becomes an abelian group. If N is also a two-sided A-module, then by defining

$$(fa)(x) = f(x)a, \quad a \in A, \quad x \in M,$$

we have $fa \in {}_A\mathbf{Mod}(M, N)$, and hence ${}_A\mathbf{Mod}(M, N)$ becomes a right A-module. When in particular $N = A$, N is a two-sided A-module, and here, the right A-module ${}_A\mathbf{Mod}(M, A)$ is called the **dual right** A**-module** of the left A-module M, which is denoted by M^*. If A is commutative, ${}_A\mathbf{Mod}(M, N)$ can be regarded as a left A-module.

EXERCISE 1.1 Given a left A-module morphism $f: M \to N$, f is an isomorphism $\Leftrightarrow$ there exists a left A-module morphism $g: N \to M$ such that $f \circ g = 1_N$ and $g \circ f = 1_M$.

EXERCISE 1.2 Let A be a commutative ring. Given left A-modules M, M', N, N' and left A-module morphisms $g: M \to M'$, $h: N \to N'$, the maps

$$g^*: {}_A\mathbf{Mod}(M', N) \to {}_A\mathbf{Mod}(M, N), \quad f \mapsto f \circ g,$$
$$h_*: {}_A\mathbf{Mod}(M, N) \to {}_A\mathbf{Mod}(M, N'), \quad f \mapsto h \circ f,$$

are left A-module morphisms.

We observe that if a subgroup N of a left A-module M satisfies the condition

$$x \in N, \quad a \in A \Rightarrow ax \in N,$$

then N is a left A-module. Such an N is called a **left** A**-submodule** of M. The factor group M/N also inherits a left A-module structure, and M/N is called a **factor left** A**-module**. Regarding a ring A as a left A-module (resp. right A-module; two-sided A-module), then an A-submodule of A is simply a **left ideal** (resp. **right ideal**; **two-sided ideal**).

Suppose now that the only left A-submodules of a left A-module M

are $\{0\}$ and M. In this situation, we call M a **simple** (or **irreducible**) **left** A**-module**. Given a left A-module morphism $f : M \to N$, the sets

$$\operatorname{Ker} f = \{x \in M; \quad f(x) = 0\},$$
$$\operatorname{Im} f = \{f(x) \in N; \quad x \in M\}$$

are left A-submodules of M and N respectively and are called the **kernel** of f and the **image** of f. The smallest left A-submodule which contains a subset S of a left A-module M is written $\langle S \rangle$ and called the left A-submodule generated by S.

Let Λ be a finite or infinite sequence of consecutive integers and let M_i $(i \in \Lambda)$ be left A-modules. Suppose we are given left A-module morphisms $f_i : M_i \to M_{i+1}$ $(i, i+1 \in \Lambda)$. When $\operatorname{Ker} f_{i+1} = \operatorname{Im} f_i$ $(i, i+1 \in \Lambda)$ for the sequence of left A-module morphisms

$$\cdots \to M_i \to M_{i+1} \to M_{i+2} \to \cdots, \tag{1.1}$$

then (1.1) is said to be an **exact sequence**. For instance, when $0 \to M \xrightarrow{f} N$ (resp. $M \xrightarrow{f} N \to 0$; $0 \to M \xrightarrow{f} N \to 0$) is an exact sequence, then f is injective (resp. surjective; bijective), and the converse also holds.

EXERCISE 1.3 Let A be a commutative ring. For a sequence $M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ of A-module morphisms to be an exact sequence, it is necessary and sufficient that, for any left A-module N, the sequence

$$0 \to {}_A\mathbf{Mod}\,(M'', N) \xrightarrow{g^*} {}_A\mathbf{Mod}\,(M, N) \xrightarrow{f^*} {}_A\mathbf{Mod}\,(M', N)$$

is exact. Furthermore, a sequence $0 \to N' \xrightarrow{f} N \xrightarrow{g} N''$ of left A-module morphisms is exact if and only if, for any left A-module M, the sequence

$$0 \to {}_A\mathbf{Mod}\,(M, N') \xrightarrow{f_*} {}_A\mathbf{Mod}\,(M, N) \xrightarrow{g_*} {}_A\mathbf{Mod}\,(M, N'')$$

is exact (cf. Exercise 1.2).

1.2 Direct products and direct sums

Let $\{M_\lambda\}_{\lambda \in \Lambda}$ be a family of left A-modules. Pick one element x_λ from each M_λ and write the resulting set $x = \{x_\lambda\}_{\lambda \in \Lambda}$, calling x_λ the λ-component of x. Let P be the set of all $x = \{x_\lambda\}_{\lambda \in \Lambda}$ constructed in the above manner. For $x=\{x_\lambda\}_{\lambda\in\Lambda}$, $y=\{y_\lambda\}_{\lambda\in\Lambda}\in P$ and $a\in A$, we define the operations

$$x+y=\{x_\lambda+y_\lambda\}_{\lambda\in\Lambda},\quad ax=\{ax_\lambda\}_{\lambda\in\Lambda},$$

which make P a left A-module. The map $p_\lambda: P\to M_\lambda$ which assigns to $x=\{x_\lambda\}_{\lambda\in\Lambda}\in P$ the λ-component x_λ of x is a left A-module morphism, and we call p_λ the canonical projection from P to M_λ. The pair $(P, \{p_\lambda\}_{\lambda\in\Lambda})$ consisting of P and the family of canonical projections p_λ ($\lambda\in\Lambda$) is called the **direct product** of the family of left A-modules $\{M_\lambda\}_{\lambda\in\Lambda}$, and is written $P=\prod_{\lambda\in\Lambda} M_\lambda$. For $\Lambda=\{1, 2, \ldots, n\}$, this is sometimes written $M_1\times\ldots\times M_n$. The direct product $(P, \{p_\lambda\}_{\lambda\in\Lambda})$ has the following property.

(P) Given a pair $(N, \{q_\lambda\}_{\lambda\in\Lambda})$ consisting of an arbitrary left A-module N and a family of left A-module morphisms $q_\lambda: N\to M_\lambda$ ($\lambda\in\Lambda$), there exists a unique left A-module morphism $f: N\to P$ which satisfies $p_\lambda\circ f=q_\lambda$ ($\lambda\in\Lambda$).

Hence the map which assigns to each $f\in{}_A\mathbf{Mod}(N, P)$, the element $\{p_\lambda\circ f\}_{\lambda\in\Lambda}\in\prod_{\lambda\in\Lambda}{}_A\mathbf{Mod}(N, M_\lambda)$ is a bijection. Furthermore, if A is a commutative ring, we have

$${}_A\mathbf{Mod}(N, P)\cong\prod_{\lambda\in\Lambda}{}_A\mathbf{Mod}(N, M_\lambda)$$

as left A-modules. The element of ${}_A\mathbf{Mod}(N, P)$ which corresponds to $\{f_\lambda\}_{\lambda\in\Lambda}\in\prod_{\lambda\in\Lambda}{}_A\mathbf{Mod}(N, M_\lambda)$ is written $\prod_{\lambda\in\Lambda} f_\lambda$ and is said to be the direct product of the A-module morphisms $\{f_\lambda\}_{\lambda\in\Lambda}$. A left A-module P with the above property is unique up to isomorphism, and the direct product of the family of left A-modules $\{M_\lambda\}_{\lambda\in\Lambda}$ is characterized by property (P).

Let S be the subset of P consisting of all those elements whose λ-components are zero except for a finite number of λs. Then S turns out to be a left A-submodule of P. When Λ is a finite set, we have $S=P$. Given $x_\lambda\in M_\lambda$, let $i_\lambda(x_\lambda)$ stand for the element of S whose λ-component is x_λ and all other components zero. Then the map $i_\lambda: M_\lambda\to S$ is a left A-module injection and is called the canonical embedding of M_λ into S. Identifying $i_\lambda(x_\lambda)$ with x_λ and regarding M_λ as a left A-submodule of S, the element $x=\{x_\lambda\}_{\lambda\in\Lambda}$ of S can be written in the form $\sum_{\lambda \in \Lambda} x_\lambda$ which, by definition, is a finite sum. The pair $(S, \{i_\lambda\}_{\lambda \in \Lambda})$ consisting of S and the family of left A-module morphisms i_λ $(\lambda \in \Lambda)$ is called the **direct sum** of the family $\{M_\lambda\}_{\lambda \in \Lambda}$ of A-modules, and is denoted $S = \coprod_{\lambda \in \Lambda} M_\lambda$ or $S = \bigoplus_{\lambda \in \Lambda} M_\lambda$. The direct sum $(S, \{i_\lambda\}_{\lambda \in \Lambda})$ has the following property.

(S) Given a pair $(N, \{j_\lambda\}_{\lambda \in \Lambda})$ consisting of an arbitary left A-module N and a family of left A-module morphisms $j_\lambda : M_\lambda \to N$, there exists a unique left A-module morphism $f : S \to N$ such that $f \circ i_\lambda = j_\lambda (\lambda \in \Lambda)$.

Thus the map that assigns to each $f \in {}_A\mathbf{Mod}(S, N)$ the element $\{f \circ i_\lambda\}_{\lambda \in \Lambda} \in \prod_{\lambda \in \Lambda} {}_A\mathbf{Mod}(M_\lambda, N)$ is a bijection. Moreover, if A is a commutative ring, this is an isomorphism

$$ {}_A\mathbf{Mod}(S, N) \cong \prod_{\lambda \in \Lambda} {}_A\mathbf{Mod}(M_\lambda, N) $$

of left A-modules. The element of ${}_A\mathbf{Mod}(S, N)$ corresponding to $\{f_\lambda\}_{\lambda \in \Lambda} \in \prod_{\lambda \in \Lambda} {}_A\mathbf{Mod}(M_\lambda, N)$ is written $\coprod_{\lambda \in \Lambda} f_\lambda$ and is called the direct sum of A-module morphisms $\{f_\lambda\}_{\lambda \in \Lambda}$. There exists a left A-module S which is unique up to isomorphism with the above property, and the direct sum of a family of left A-modules $\{M_\lambda\}_{\lambda \in \Lambda}$ can be characterized as the left A-module which satisfies property (S).

Free A-modules Let A be a ring. Given a set Λ, we assign to each $\lambda \in \Lambda$ a left A-module A_λ which is isomorphic to A, and denote the direct sum of the family of left A-modules $\{A_\lambda\}_{\lambda \in \Lambda}$ by $F_A(\Lambda) = \coprod_{\lambda \in \Lambda} A_\lambda$. If we identify the image under i_λ of the identity element 1_λ of A_λ with λ, then we have $\Lambda \subset F_A(\Lambda)$, and an element x of $F_A(\Lambda)$ can be written uniquely in the form $x = \sum_{\lambda \in \Lambda} x_\lambda \lambda$ where $x_\lambda \in A$ and $x_\lambda = 0$ except for a finite number of λ. In this situation, $F_A(\Lambda)$ is said to be the **free left A-module** generated by Λ. Letting Map (Λ, M) be the family of all maps from Λ to a left A-module M, we obtain a one-to-one correspondence

$$ {}_A\mathbf{Mod}(F_A(\Lambda), M) \cong \mathrm{Map}(\Lambda, M). $$

Now, Map (Λ, M) has a left A-module structure. Moreover, when A is a commutative ring, the two left A-modules above are isomorphic as left A-modules under the same correspondence.

Bases Let A be a ring and S a subset of a left A-module M. If S satisfies conditions (1) and (2) below, S is said to be a **basis** for the A-module M.

(1) For any finite subset $\{s_1, \ldots, s_n\}$ of S,

$$\sum_{i=1}^{n} a_i s_i = 0, \quad a_i \in A \ (1 \leqq i \leqq n) \Rightarrow a_i = 0 \ (1 \leqq i \leqq n).$$

(2) Given an arbitrary element $x \in M$, a finite subset $\{s_1, \ldots, s_n\}$ of S can be chosen appropriately so that x can be written in the form

$$x = \sum_{i=1}^{n} a_i s_i \quad \text{where} \quad a_i \in A \ (1 \leqq i \leqq n).$$

A set which satisfies condition (1) is called a **linearly independent** set over A, and a set which satisfies condition (2) is called a **system of generators** of M over A. In particular, a left A-module which has a finite system of generators is said to be **finitely generated**. If $F_A(\Lambda)$ is a free A-module over Λ, Λ becomes a basis for $F_A(\Lambda)$ when we regard Λ as a subset of $F_A(\Lambda)$. Conversely, an A-module which has a basis S is isomorphic to the free A-module $F_A(S)$. Consequently, for a left A-module to be a free left A-module, it is necessary and sufficient that it have a basis. Moreover, when A is a commutative ring, the number of elements in a basis for a finitely generated free A-module is constant regardless of the choice of a basis. This number is called the **rank** of the free A-module. Given a field k, a k-vector space has a basis, and is therefore a free k-module. We call the rank of a finitely generated k-vector space V its **dimension**, and write dim V.

EXERCISE 1.4 Let V_1 and V_2 be subspaces of a finite dimensional k-vector space V. Then

(i) $V_1 \subset V_2 \Rightarrow \dim V_1 \leqq \dim V_2$.

(ii) $V_1 \subset V_2$ and $\dim V_1 = \dim V_2 \Rightarrow V_1 = V_2$.

(iii) $\dim V_1 + \dim V_2 = \dim(V_1 + V_2) + \dim(V_1 \cap V_2)$, where $V_1 + V_2$ is the subspace of V generated by $V_1 \cap V_2$.

EXERCISE 1.5 If $f: V \to V'$ is a k-linear map, then $\dim V = \dim(\operatorname{Im} f) + \dim(\operatorname{Ker} f)$.

Completely reducible modules Let M be a left A-module. Given any left A-submodule N of M, if there exists a left A-submodule N' of M such that $M = N \oplus N'$ (the direct sum of N and N'), then M is said to be **completely reducible**. A completely reducible left A-module can be expressed as the direct sum of its irreducible left A-submodules. Further, a left A-submodule of a completely reducible left A-module is also completely reducible. (See for instance Hattori [1], Theorems 15.7, 15.8, or Curtis–Reiner [5], 15.2, 15.3.)

1.3 Tensor products

Suppose we are given a ring A, a right A-module M, and a left A-module N. Let $F(M \times N)$ be the free $\mathbb{Z}$-module generated by the set

$$M \times N = \{(x, y); \quad x \in M, \quad y \in N\}$$

and let $K(M \times N)$ be the $\mathbb{Z}$-submodule of $F(M \times N)$ generated by all elements of the type

$$(x + x', y) - (x, y) - (x', y),$$
$$(x, y + y') - (x, y) - (x, y'),$$
$$(xa, y) - (x, ay),$$

for all $x, x' \in M$, $y, y' \in N$, $a \in A$. In these circumstances, we call the factor group $F(M \times N)/K(M \times N)$ the **tensor product** of M and N over A, and denote it by $M \otimes_A N$ or simply by $M \otimes N$. The residue class which contains (x, y) is written $x \otimes y$. For $x, x' \in M$, $y, y' \in N$, $a \in A$, we have by definition

$$(x + x') \otimes y = x \otimes y + x' \otimes y,$$
$$x \otimes (y + y') = x \otimes y + x \otimes y',$$
$$xa \otimes y = x \otimes ay.$$

An element of $M \otimes_A N$ can be written in the form $\sum_{i=1}^{n} x_i \otimes y_i$ where $x_i \in M$, $y_i \in N$. If M is a two-sided (B, A)-module (resp. N is a two-sided

(A, C)-module), then the definition

$$b(x \otimes y) = bx \otimes y, \quad b \in B, x \in M, y \in N$$
$$(\text{resp.} \quad (x \otimes y)c = x \otimes yc, \quad c \in C, x \in M, y \in N),$$

makes $M \otimes_A N$ a left B-module (resp. right C-module). When A is a commutative ring, we can define

$$a(x \otimes y) = ax \otimes y = x \otimes ay, \quad a \in A, \quad x \in M, \quad y \in N,$$

which makes $M \otimes_A N$ an A-module.

Bilinear maps Let A be a commutative ring and let M, N, T be A-modules. A map $f : M \times N \to T$ such that

$$f(ax + bx', y) = af(x, y) + bf(x', y),$$
$$f(x, ay + by') = af(x, y) + bf(x, y'),$$
$$a, b \in A, \quad x, x' \in M, \quad y, y' \in N,$$

is called a **bilinear map**. The set of all bilinear maps from $M \times N$ to T is denoted $B_A(M \times N, T)$. By the definition of tensor products, the map from $M \times N$ to the A-module $M \otimes_A N$

$$\varphi : M \times N \to M \otimes_A N$$

given by $(x, y) \mapsto x \otimes y$ is bilinear, and is called the **canonical bilinear map**. Suppose we are given an A-module T and an A-module morphism $g : M \otimes_A N \to T$. Setting $f = g \circ \varphi$, we see that $f : M \times N \to T$ is a bilinear map. Moreover, the map

$$\Phi : {}_A\mathbf{Mod}\,(M \otimes_A N, T) \to B_A(M \times N, T)$$

which carries g to f is a bijection. In fact, for $f \in B_A(M \times N, T)$, we define $g(x \otimes y) = f(x, y)$, thereby obtaining an A-module morphism $g : M \otimes_A N \to T$. The map defined by $f \mapsto g$ turns out to be the inverse of Φ. The set $B_A(M \times N, T)$ admits an A-module structure making Φ an A-module isomorphism. Further, for any A-module T, the tensor product $M \otimes_A N$ which makes Φ a bijection is unique up to isomorphism. Moreover, the tensor product $M \otimes_A N$ is characterized by the above mentioned property. When M is a free A-module with a basis $\{e_\lambda\}_{\lambda \in \Lambda}$, an element of $M \otimes_A N$ can be written uniquely in the form $\sum_{\lambda \in \Lambda} e_\lambda \otimes y_\lambda$ where $y_\lambda \in N$, $\lambda \in \Lambda$ and $y_\lambda = 0$ except for a finite number of λs. Furthermore, if M and N are free A-modules, so is $M \otimes_A N$.

EXERCISE 1.6 Given a commutative ring A and A-modules M, N, P, prove that

(i) $A \otimes_A M \cong M$,

(ii) $M \otimes_A N \cong N \otimes_A M$,

(iii) $(M \otimes_A N) \otimes_A P \cong M \otimes_A (N \otimes_A P)$.

EXERCISE 1.7 Given a commutative ring A and A-modules M, N, let M^*, N^* be the dual A-modules of M, N respectively. Show that

$$_A\mathbf{Mod}\,(M, N^*) \cong {}_A\mathbf{Mod}\,(N, M^*) \cong B_A(M \times N, A).$$

Remark The map $F : M \mapsto M^*$ from the category $_A\mathbf{Mod}$ to itself is a contravariant functor, and F is adjoint to itself.

EXERCISE 1.8 Given a field k, let M, N be k-vector spaces with dual k-vector spaces M^*, N^* respectively.

(i) Define a map $\varphi : M^* \otimes N \to \mathbf{Mod}_k(M, N)$ for $f \in M^*$, $y \in N$, $x \in M$ by $\varphi(f \otimes y)(x) = f(x)y$. Then φ is a k-linear injection. Moreover, φ is a k-linear isomorphism if M or N is finite dimensional.

(ii) Define a map $\rho : M^* \otimes N^* \to (M \otimes N)^*$ for $f \in M^*$, $g \in N^*$, $x \in M$, $y \in N$ by $\rho(f \otimes g)\ (x \otimes y) = f(x)g(y)$. Then ρ is a k-linear injection, and is furthermore a k-linear isomorphism when both M and N are finite dimensional.

EXERCISE 1.9 Let V, V' be k-vector spaces, and let W, W' be subspaces of V, V' respectively. Then

(i) The canonical embedding $W \otimes W' \to V \otimes V'$ is a k-linear injection.

(ii) $(V \otimes W') \cap (W \otimes V') = W \otimes W'$.

(iii) The kernel of the canonical projection $f : V \otimes V' \to V/W \otimes V'/W'$ is given by

$$\operatorname{Ker} f = V \otimes W' + W \otimes V'.$$

Change of rings of scalars Let A, B be rings and F a two-sided