1. *Matrices and determinants*

General remarks

Throughout Chapter 1, R will denote a commutative ring with an identity element. Usually the identity element of R will be denoted by 1, but 1_R will be used if we wish to draw attention to the ring in which we are working. This chapter is devoted to reviewing certain well known basic facts concerning matrices and determinants and organizing them in a way which will be useful later.

1.1 Matrices

By a *matrix with entries in R*, or an *R-matrix*, it is customary to understand a rectangular array of elements taken from the ring R. However there are many situations where the order of the rows and the order of the columns is not important. Again, in the theory of vector spaces, a linear mapping of one finite dimensional vector space into another can be described by means of a matrix as soon as each of the two spaces involved has been provided with a base. Now it can happen that one or possibly both of the spaces in question has dimension zero. To deal with such situations we need, for example, the notion of a matrix which has p rows (say) and zero columns. It is because of considerations such as these that we make a fresh start and approach the idea of a matrix from a slightly more general standpoint.

Let M and N be finite sets. By an $M \times N$ *matrix* with entries in R we shall mean a mapping

$$A: M \times N \to R \tag{1.1.1}$$

of the cartesian product $M \times N$ into the ring R. Suppose that we have such a matrix. Let $m \in M$ and $n \in N$. Then (m, n) has an image in R which we may denote by a_{mn}. As an alternative notation for the matrix A we shall use

$$A = \|a_{mn}\| \quad (m \in M, n \in N). \tag{1.1.2}$$

This will be abbreviated to $A = \|a_{mn}\|$ if there is no risk of confusion.

Given an $M \times N$ matrix $A = \|a_{mn}\|$ we can associate with it an $N \times M$ matrix $C = \|c_{nm}\|$ by requiring that $c_{nm} = a_{mn}$ for all $m \in M$ and $n \in N$. The matrix C is called the *transpose* of A and we shall indicate the connection between the two by writing $C = A^{\mathrm{T}}$. Evidently if C is the transpose of A, then A is the transpose of C.

Now let M, N and Q be finite sets. Suppose that $A = \|a_{mn}\|$ is an $M \times N$ matrix and $B = \|b_{nq}\|$ is an $N \times Q$ matrix. This time we define an $M \times Q$ matrix $C = \|c_{mq}\|$ by putting

$$c_{mq} = \sum_{n \in N} a_{mn} b_{nq} \tag{1.1.3}$$

when $m \in M$ and $q \in Q$. The matrix C is called the *product* of A and B and we write $C = AB$. Clearly

$$(AB)^{\mathrm{T}} = B^{\mathrm{T}} A^{\mathrm{T}}. \tag{1.1.4}$$

Note that the possibility that N may be the empty set is not excluded. In such a situation the product AB will be the zero $M \times Q$ matrix, that is to say each of its entries will be the zero element of R.

Let p and q be positive integers and take $M = \{1, 2, \ldots, p\}$, $N = \{1, 2, \ldots, q\}$. For these particular choices of M and N it is customary to refer to an $M \times N$ matrix as a $p \times q$ matrix and we shall observe this custom. Also if $A = \|a_{jk}\|$, where $1 \leqslant j \leqslant p$ and $1 \leqslant k \leqslant q$, is a $p \times q$ matrix we shall, when convenient, exhibit the relation between A and its individual entries by writing

$$A = \left\| \begin{matrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1q} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2q} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{p1} & a_{p2} & a_{p3} & \cdots & a_{pq} \end{matrix} \right\|.$$

In fact most of our matrices will be presented in this way and it is only in a few (but important) situations where the extra generality will prove useful.

1.2 Determinants

Let M be a finite set. An $M \times M$ matrix will be called a *square matrix of type M*. The product of two such matrices will again be a square matrix of type M. Note that this multiplication is associative but not in general commutative. Furthermore the *identity matrix*, that is the matrix

$$I_M = \|\delta_{\mu m}\| \quad (\mu, m \in M), \tag{1.2.1}$$

where $\delta_{\mu m}$ is zero if $\mu \neq m$ and is 1_R otherwise, is neutral with respect to multiplication.

Let $A = \|a_{\mu m}\|$ be a square matrix of type M and let π be a permutation of the set M. The *determinant* of A, which we denote by $\det(A)$ or $|A|$, is defined by

$$\det(A) = \sum_{\pi} \operatorname{sgn}(\pi) \left(\prod_{\mu \in M} a_{\mu\pi(\mu)} \right). \tag{1.2.2}$$

Here $\operatorname{sgn}(\pi) = +1$ if π is an even permutation and $\operatorname{sgn}(\pi) = -1$ if it is an odd permutation, i.e. $\operatorname{sgn}(\pi)$ is the *signature* or *parity* of π. In this connection it is convenient to have a convention to cover the case where M is the empty set, that is where we have to do with a square matrix with zero rows and zero columns. In fact we shall define the determinant in this case to be the identity element of R. Thus symbolically

$$\det(\|\cdot\|) = 1_R. \tag{1.2.3}$$

It has already been remarked that the square matrices of type M form a system which is closed under multiplication and which has the matrix $I_M = \|\delta_{\mu m}\|$ as identity element. The system is also closed with respect to the process of replacing a matrix by its transpose. This structure has a familiar connection with the theory of determinants. Thus if A, B are square matrices of type M, then

$$\det(AB) = \det(A)\det(B), \tag{1.2.4}$$

$$\det(I_M) = 1_R, \tag{1.2.5}$$

and

$$\det(A^{\mathrm{T}}) = \det(A). \tag{1.2.6}$$

An $M \times M$ matrix A is called *invertible* if there exists an $M \times M$ matrix C such that $AC = I_M = CA$. (Of course if C exists, then it is unique.) C is called the *inverse* of A and it is denoted by A^{-1}. It is well known that A is invertible if and only if $\det(A)$ is a unit of the ring R. Invertible matrices are also known as *unimodular* matrices.

In the three exercises which follow p and q denote positive integers.

EXERCISE 1.† *Let B be a $p \times p$ matrix (with entries in R) and $c_1, c_2, \ldots, c_p$ elements of R such that $(c_1, c_2, \ldots, c_p) B = 0$. Show that $c_i \det(B) = 0$ for $i = 1, 2, \ldots, p$.*

EXERCISE 2. *Suppose that $AB = A$, where $A = \|a_{jk}\|$ is a $p \times q$ and $B = \|b_{km}\|$ a $q \times q$ matrix with entries in R. Let $\mathfrak{A}$ be the ideal generated by the a_{jk} and $\mathfrak{B}$ the ideal generated by the b_{km}. Show that there exists $\beta \in \mathfrak{B}$ such that $(1-\beta)\mathfrak{A} = 0$.*

We recall that an element e, of R, is called an *idempotent* if $e^2 = e$. Both the zero element and the identity element are idempotents. An idempotent which is different from these is called a *non-trivial* idempotent. An integral domain, for example, has no non-trivial idempotents.

EXERCISE 3. *Suppose that $A\Omega A = A$, where A is a $p \times q$ matrix and Ω a $q \times p$ matrix. Let $\mathfrak{A}$ be the ideal generated by the entries in A. Show that there exists an idempotent α such that $\mathfrak{A} = R\alpha$. (Hence $(1-\alpha)\mathfrak{A} = 0$ and if R has no non-trivial idempotents, then either $\mathfrak{A} = 0$ or $\mathfrak{A} = R$.)*

1.3 The exterior powers of a matrix

Throughout section (1.3) the letters p, q and t will denote positive integers. Suppose that $\nu \geq 0$ is an integer. We shall denote by S_ν^p the set of all sequences $J = \{j_1, j_2, \ldots, j_\nu\}$, where $1 \leq j_1 < j_2 < \ldots < j_\nu \leq p$. Evidently S_ν^p contains $\binom{p}{\nu}$ members. Thus S_ν^p is empty when $\nu > p$ and is non-empty in all other cases. In particular it contains a single member when $\nu = 0$.

† Solutions to the exercises will be found at the end of the chapter.

Let $A = \|a_{jk}\|$ be a $p \times q$ matrix with entries in R. Suppose, for the moment, that $1 \leqslant \nu \leqslant \min(p, q)$. If now $J = \{j_1, j_2, \ldots, j_\nu\}$ belongs to S_ν^p and $K = \{k_1, k_2, \ldots, k_\nu\}$ to S_ν^q, then we put

$$A_{JK}^{(\nu)} = \begin{Vmatrix} a_{j_1 k_1} & a_{j_1 k_2} & \cdots & a_{j_1 k_\nu} \\ a_{j_2 k_1} & a_{j_2 k_2} & \cdots & a_{j_2 k_\nu} \\ \vdots & \vdots & \cdots & \vdots \\ a_{j_\nu k_1} & a_{j_\nu k_2} & \cdots & a_{j_\nu k_\nu} \end{Vmatrix} \tag{1.3.1}$$

so that $A_{JK}^{(\nu)}$ is a typical $\nu \times \nu$ minor of A. Let us keep ν fixed. Then the $A_{JK}^{(\nu)}$, where $J \in S_\nu^p$ and $K \in S_\nu^q$, may be regarded as the entries in an $S_\nu^p \times S_\nu^q$ matrix $A^{(\nu)}$. This matrix $A^{(\nu)}$ is called the νth *exterior power* of A. At this point we relax the restriction that was placed on ν and regard $A^{(\nu)}$ as being defined for all $\nu \geqslant 0$. Note that, by (1.2.3), $A^{(0)}$ is the 1×1 matrix $\|1_R\|$ and that $A^{(1)} = A$. Note also that *if C is the transpose of A, then $C^{(\nu)}$ is the transpose of $A^{(\nu)}$.* Again, *the exterior powers of an identity matrix are themselves identity matrices*. In this context a matrix with zero rows and zero columns counts as an identity matrix.†

Let $A = \|a_{jk}\|$ be a $p \times q$ matrix and $B = \|b_{km}\|$ a $q \times t$ matrix. Put $C = AB$, say $C = \|c_{jm}\|$, and suppose for the moment that $1 \leqslant \nu \leqslant \min(p, t)$. If now $J = \{j_1, j_2, \ldots, j_\nu\}$ belongs to S_ν^p and $M = \{m_1, m_2, \ldots, m_\nu\}$ to S_ν^t, then

$$C_{JM}^{(\nu)} = \begin{vmatrix} c_{j_1 m_1} & c_{j_1 m_2} & \cdots & c_{j_1 m_\nu} \\ c_{j_2 m_1} & c_{j_2 m_2} & \cdots & c_{j_2 m_\nu} \\ \vdots & \vdots & \cdots & \vdots \\ c_{j_\nu m_1} & c_{j_\nu m_2} & \cdots & c_{j_\nu m_\nu} \end{vmatrix}$$

$$= \begin{vmatrix} \sum\limits_\alpha a_{j_1\alpha} b_{\alpha m_1} & \sum\limits_\beta a_{j_1\beta} b_{\beta m_2} & \cdots & \sum\limits_\gamma a_{j_1\gamma} b_{\gamma m_\nu} \\ \sum\limits_\alpha a_{j_2\alpha} b_{\alpha m_1} & \sum\limits_\beta a_{j_2\beta} b_{\beta m_2} & \cdots & \sum\limits_\gamma a_{j_2\gamma} b_{\gamma m_\nu} \\ \vdots & \vdots & \cdots & \vdots \\ \sum\limits_\alpha a_{j_\nu\alpha} b_{\alpha m_1} & \sum\limits_\beta a_{j_\nu\beta} b_{\beta m_2} & \cdots & \sum\limits_\gamma a_{j_\nu\gamma} b_{\gamma m_\nu} \end{vmatrix},$$

whence

$$C_{JM}^{(\nu)} = \sum_{\alpha, \beta, \ldots, \gamma} b_{\alpha m_1} b_{\beta m_2} \cdots b_{\gamma m_\nu} \begin{vmatrix} a_{j_1\alpha} & a_{j_1\beta} & \cdots & a_{j_1\gamma} \\ a_{j_2\alpha} & a_{j_2\beta} & \cdots & a_{j_2\gamma} \\ \vdots & \vdots & \cdots & \vdots \\ a_{j_\nu\alpha} & a_{j_\nu\beta} & \cdots & a_{j_\nu\gamma} \end{vmatrix}. \tag{1.3.2}$$

† Cf. (1.2.3).

Here $\alpha, \beta, \ldots, \gamma$ range freely between 1 and q. However, unless they are distinct, the determinant in (1.3.2) vanishes. Let $K = \{k_1, k_2, \ldots, k_\nu\}$ belong to S_ν^q. If $\{\alpha, \beta, \ldots, \gamma\}$ is a permutation of $\{k_1, k_2, \ldots, k_\nu\}$, then

$$\begin{vmatrix} a_{j_1\alpha} & a_{j_1\beta} & \cdots & a_{j_1\gamma} \\ a_{j_2\alpha} & a_{j_2\beta} & \cdots & a_{j_2\gamma} \\ \vdots & \vdots & \cdots & \vdots \\ a_{j_\nu\alpha} & a_{j_\nu\beta} & \cdots & a_{j_\nu\gamma} \end{vmatrix} = \epsilon_{\alpha\beta\ldots\gamma} A_{JK}^{(\nu)},$$

where $\epsilon_{\alpha\beta\ldots\gamma}$ has the value $+1$ or -1 according as $\{\alpha, \beta, \ldots, \gamma\}$ is an even or an odd permutation of $\{k_1, k_2, \ldots, k_\nu\}$. It follows that the contribution to the sum on the right hand side of (1.3.2) from *all* the permutations of $\{k_1, k_2, \ldots, k_\nu\}$ is $A_{JK}^{(\nu)} B_{KM}^{(\nu)}$. Accordingly

$$C_{JM}^{(\nu)} = \sum_K A_{JK}^{(\nu)} B_{KM}^{(\nu)} \tag{1.3.3}$$

and we have proved

THEOREM 1. *Let A be a $p \times q$ matrix and B a $q \times t$ matrix. Then $(AB)^{(\nu)} = A^{(\nu)} B^{(\nu)}$ for every $\nu \geqslant 0$.*

Note that, because of our conventions, we can allow ν to be any non-negative integer.

We recall that R is called a *non-trivial* ring if its identity element is not zero.

EXERCISE 4. *Suppose that the ring R is non-trivial. Let A be a $p \times q$ matrix and B a $q \times p$ matrix such that AB and BA are both of them identity matrices. Show that $p = q$. (Thus A and B are unimodular matrices and each is the inverse of the other.)*

1.4 Determinantal ideals

Let A be a $p \times q$ matrix and B a $q \times t$ matrix, where p, q, t are positive integers. If now $\nu \geqslant 0$ is an integer, denote by $\mathfrak{A}_\nu(A)$ the ideal generated by the entries in $A^{(\nu)}$. Thus $\mathfrak{A}_\nu(A)$ is the ideal generated by all the $\nu \times \nu$ minors of A.

DEFINITION. *The ideals $\mathfrak{A}_\nu(A)$, where $\nu = 0, 1, 2, \ldots$, are called the 'determinantal ideals' of A.*

Since $A^{(0)} = \|1_R\|$, it follows that $\mathfrak{A}_0(A) = R$. Of course $\mathfrak{A}_1(A)$ is the ideal generated by the elements of A. Again

$$\mathfrak{A}_\nu(A) = 0 \quad \text{for} \quad \nu > \min(p, q) \tag{1.4.1}$$

and $$R = \mathfrak{A}_0(A) \supseteq \mathfrak{A}_1(A) \supseteq \mathfrak{A}_2(A) \supseteq \mathfrak{A}_3(A) \supseteq \dots. \quad (1.4.2)$$

Also, because $(AB)^{(\nu)} = A^{(\nu)}B^{(\nu)}$ by Theorem 1, we have

$$\mathfrak{A}_\nu(AB) \subseteq \mathfrak{A}_\nu(A) \cap \mathfrak{A}_\nu(B) \quad (1.4.3)$$

and this extends to a product of any finite number of matrices.

THEOREM 2. *Let A and A' be $p \times q$ matrices and suppose that $A' = UAV$, $A = U'A'V'$, where U, U' are $p \times p$ matrices and V, V' are $q \times q$ matrices. Then $\mathfrak{A}_\nu(A) = \mathfrak{A}_\nu(A')$ for all $\nu \geqslant 0$.*

Proof. Since $A' = UAV$, we have $\mathfrak{A}_\nu(A') \subseteq \mathfrak{A}_\nu(A)$ by (1.4.3) and the opposite inclusion holds similarly.

DEFINITION. *The $p \times q$ matrix A' is said to be 'equivalent' to the $p \times q$ matrix A if there exist unimodular matrices U, V such that $A' = UAV$.*

This relation is reflexive, symmetric, and transitive. Indeed if $A' = UAV$, where U, V are unimodular and therefore invertible, then $A = U^{-1}A'V^{-1}$. We can therefore apply Theorem 2 and so obtain

THEOREM 3. *If A and A' are equivalent $p \times q$ matrices, then $\mathfrak{A}_\nu(A) = \mathfrak{A}_\nu(A')$ for all $\nu \geqslant 0$.*

It is a classical result that if R is an integral domain with the property that every ideal can be generated by a single element, then the converse of Theorem 3 holds. Thus for such an integral domain two $p \times q$ matrices A and A' are equivalent if and only if $\mathfrak{A}_\nu(A) = \mathfrak{A}_\nu(A')$ for all $\nu \geqslant 0$. As we shall not be making use of this result we refer the interested reader to the literature.†

We recall that by *elementary row and column operations on A* it is customary to mean the following:

(1) multiplication of the elements of any row or column of A by one and the same unit;

(2) interchanging any two rows or columns of A;

(3) adding to any row (column) of A a multiple, by an element of R, of a different row (column).

† See (6) in the list of references at the end. The section dealing with these matters is Chapter 7, §4, no. 5.

If we take any one of these operations, then the same effect can be produced by multiplying A by a suitable unimodular matrix. For example suppose that $1 \leqslant i, j \leqslant p$ and $i \neq j$. Denote by U the matrix produced by taking the identity matrix of order p and putting an element α, where $\alpha \in R$, in the (i,j)th position. Then $\det(U) = 1_R$, so U is unimodular, and UA is the matrix one obtains from A by adding α times the jth row to the ith row. Accordingly we have

THEOREM 4. *Let the matrix A' be obtained from A by means of elementary row and column operations. Then A and A' are equivalent and hence they have the same determinantal ideals.*

We now give a partial result concerning determinantal ideals.

LEMMA 1.† *Suppose that $A\Omega A = A$, where A is a $p \times q$ matrix and Ω a $q \times p$ matrix. Then for each $\nu \geqslant 0$ the determinantal ideal $\mathfrak{A}_\nu(A)$ is generated by an idempotent. Hence if R has no non-trivial idempotents, then either $\mathfrak{A}_\nu(A) = 0$ or $\mathfrak{A}_\nu(A) = R$.*

Proof. We may confine our attention to the case where $1 \leqslant \nu \leqslant \min(p, q)$. By Theorem 1, $A^{(\nu)}\Omega^{(\nu)}A^{(\nu)} = A^{(\nu)}$ and by definition $\mathfrak{A}_\nu(A)$ is the ideal generated by the entries in $A^{(\nu)}$. The desired result therefore follows from Exercise 3.

1.5 Some useful formulae

Throughout section (1.5) we shall be concerned with a $p \times q$ matrix $A = \|a_{jk}\|$, where p, q are positive integers. Suppose that $0 \leqslant \mu \leqslant \min(p-1, q)$ and let $M = \{m_1, m_2, \ldots, m_{\mu+1}\}$ belong to $S^p_{\mu+1}$ and $K = \{k_1, k_2, \ldots, k_\mu\}$ to S^q_μ, where the notation is as explained in section (1.3). We now define a row vector x_{MK} of length p by

$$(x_{MK})_j = \begin{cases} 0 & \text{if } j \notin M, \\ (-1)^{\alpha+1} A^{(\mu)}_{M \setminus j, K} & \text{if } j = m_\alpha. \end{cases} \tag{1.5.1}$$

† The converse of the first assertion of the lemma is also true. Cf. Chapter 4 Theorem 18.

Here by $M\backslash m_\alpha$ we mean M with the term m_α removed. Since x_{MK} is a row vector of length p we can form $x_{MK}A$ and this will be a row vector of length q. In fact for $1 \leqslant k \leqslant q$ we have

$$(x_{MK}A)_k = \begin{vmatrix} a_{m_1 k} & a_{m_1 k_1} & \cdots & a_{m_1 k_\mu} \\ a_{m_2 k} & a_{m_2 k_1} & \cdots & a_{m_2 k_\mu} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m_{\mu+1} k} & a_{m_{\mu+1} k_1} & \cdots & a_{m_{\mu+1} k_\mu} \end{vmatrix} \qquad (1.5.2)$$

as is readily verified by expanding the determinant by means of its first column.

There is a natural companion to this result. To describe it suppose that $0 \leqslant \nu \leqslant \min(p, q-1)$. Further suppose that $J = \{j_1, j_2, \ldots, j_\nu\}$ belongs to S^p_ν and $N = \{n_1, n_2, \ldots, n_{\nu+1}\}$ to $S^q_{\nu+1}$. We can now define a column vector y_{JN} of length q by

$$(y_{JN})_k = \begin{cases} 0 & \text{if} \quad k \notin N, \\ (-1)^{\beta+1} A^{(\nu)}_{J, N\backslash k} & \text{if} \quad k = n_\beta. \end{cases} \qquad (1.5.3)$$

This secures that Ay_{JN} is a column vector of length p and we have

$$(Ay_{JN})_j = \begin{vmatrix} a_{jn_1} & a_{jn_2} & \cdots & a_{jn_{\nu+1}} \\ a_{j_1 n_1} & a_{j_1 n_2} & \cdots & a_{j_1 n_{\nu+1}} \\ \vdots & \vdots & \cdots & \vdots \\ a_{j_\nu n_1} & a_{j_\nu n_2} & \cdots & a_{j_\nu n_{\nu+1}} \end{vmatrix}. \qquad (1.5.4)$$

On this occasion the relation can be checked by expanding the determinant on the right hand side of (1.5.4) by means of its first row.

Our next result is somewhat more complicated in character and in order to present it we shall require some additional notation. Let $t \geqslant 0$ be an integer and let $H = \{h_1, h_2, \ldots, h_t\}$. This is to be a sequence of integers between 1 and p, but on this occasion we do not postulate that $h_1, h_2, \ldots, h_t$ be distinct nor do we insist that they should form an increasing sequence. Suppose next that $1 \leqslant \mu \leqslant p$. We put

$$\omega(\mu, H) = \begin{cases} 0 & \text{if} \quad H \text{ contains a repetition or } \mu \notin H, \\ (-1)^\alpha & \text{if} \quad H \text{ contains no repetitions and} \quad \mu = h_\alpha. \end{cases} \qquad (1.5.5)$$

Again let $L = \{l_1, l_2, \ldots, l_t\}$ also be a sequence of integers but this time between 1 and q, where once more repetitions are allowed. For $1 \leqslant \nu \leqslant q$ we define $\omega(\nu, L)$ by means of a formula analogous to (1.5.5).

After these preliminaries put

$$\Delta_{HL} = \begin{vmatrix} a_{h_1 l_1} & a_{h_1 l_2} & \cdots & a_{h_1 l_t} \\ a_{h_2 l_1} & a_{h_2 l_2} & \cdots & a_{h_2 l_t} \\ \vdots & \vdots & \cdots & \vdots \\ a_{h_t l_1} & a_{h_t l_2} & \cdots & a_{h_t l_t} \end{vmatrix} \tag{1.5.6}$$

and for $1 \leqslant h \leqslant p$, $1 \leqslant l \leqslant q$ set

$$hH = \{h, h_1, h_2, \ldots, h_t\} \quad \text{and} \quad lL = \{l, l_1, l_2, \ldots, l_t\}. \tag{1.5.7}$$

Using these we can define a $p \times q$ matrix Θ_{HL} by

$$(\Theta_{HL})_{hl} = \Delta_{hH, lL}. \tag{1.5.8}$$

We can also define a $q \times p$ matrix Ω_{HL} as follows: if either H or L contains a repetition, then $\Omega_{HL} = 0$; on the other hand if neither H nor L contains a repetition, then the entries in Ω_{HL} are given by

$$(\Omega_{HL})_{\nu\mu} = \begin{cases} 0 & \text{if either} \quad \mu \notin H \quad \text{or} \quad \nu \notin L, \\ \omega(\mu, H)\, \omega(\nu, L)\, \Delta_{H\backslash\mu,\, L\backslash\nu} & \text{if} \quad \mu \in H \quad \text{and} \quad \nu \in L. \end{cases}$$

The significance of these various definitions is revealed by

THEOREM 5. *Let the notation be as above. Then*

$$\Theta_{HL} = \Delta_{HL} A - A \Omega_{HL} A. \tag{1.5.9}$$

Consequently if all the $(t+1) \times (t+1)$ *minors of* A *are zero, that is if* $\mathfrak{A}_{t+1}(A) = 0$, *then* $\Delta_{HL} A = A \Omega_{HL} A$.

Proof. We may suppose that neither H nor L contains a repetition. Now

$$(\Theta_{HL})_{hl} = \begin{vmatrix} a_{hl} & a_{hl_1} & \cdots & a_{hl_t} \\ a_{h_1 l} & a_{h_1 l_1} & \cdots & a_{h_1 l_t} \\ \vdots & \vdots & \cdots & \vdots \\ a_{h_t l} & a_{h_t l_1} & \cdots & a_{h_t l_t} \end{vmatrix}.$$