CHAPTER I

FORMAL PRELIMINARIES

1. The summation convention.

The theory of differential invariants, like the theory of algebraic invariants, is essentially formal. It has many geometrical and physical applications which have played a large rôle in the development of the theory. Yet, after all, it is the actual formulas which are the essential subject matter of the theory. This, at least, is the point of view which we are adopting at present, and so we shall devote a chapter to questions of notation before we try to say what a differential invariant is.

Recent advances in the theory of differential invariants and the wide use of this theory in physical investigations have brought about a rather general acceptance of a particular type of notation, the essential feature of which is the systematic use of subscripts and superscripts and the resulting abandonment of all sorts of notations for special operations. The only operations for which we shall employ special signs are addition, subtraction, multiplication, division, differentiation and integration. These, with the usual run of symbols for functions and sets of functions, are found to suffice for all our purposes.

We shall follow the usage of Einstein of indicating a summation by means of a repeated index: i.e. any term in which the same index (subscript or superscript) appears twice shall stand for the sum of all terms obtainable by giving the index all possible values. Thus, for example, if i can take on the values* from 1 to n,

$$a_1x^1 + a_2x^2 + \dots + a_nx^n = a_ix^i. \tag{1·1}$$

Before the advent of Relativity, this expression would have been written

$$\sum_{i=1}^{n} a_ix^i.$$

The innovation consists simply in leaving off the summation sign whenever an index is repeated. Its only inconvenience arises when

* The superscript k in x^k does not mean that x is raised to the power k but is merely an index to distinguish among n variables, $x^1, x^2, \dots, x^n$.

we wish to speak of the general term in an expression like (1·1) without carrying out the summation. But in the theory of differential invariants this situation arises so rarely that the inconvenience is negligible in comparison with the advantages of the notation.

The repeated index is sometimes called a *dummy* or an *umbral* index because, like a variable of integration, the symbol for it in any expression can be changed without affecting the meaning of the expression. Thus

$$a_i x^i = a_k x^k.$$

We shall have to deal with sets of quantities

$$T^{ab\ldots c}_{ij\ldots k} \tag{1·2}$$

which are in general functions of n variables $x^1, x^2, \ldots, x^n$. If there are p superscripts and q subscripts each taking values from 1 to n, the expression (1·2) indicates a set of n^{p+q} quantities. By setting a subscript and a superscript equal to each other and summing according to the summation convention we can get a new set of quantities, for example,

$$P^{b\ldots c}_{j\ldots k} = T^{ab\ldots c}_{aj\ldots k}.$$

This operation is called *contraction* (German, *Verjüngung*).

When we have two sets of quantities and multiply every quantity in one set by every quantity in the other set, for example,

$$P^{ab\ldots c}_{ij\ldots k}\, Q^{de\ldots f}_{lm\ldots p} = R^{ab\ldots cde\ldots f}_{ij\ldots klm\ldots p},$$

we get a new and more numerous set of quantities of the same type. This operation is called *multiplication*. When we have two sets of quantities indicated by the same numbers of subscripts and of superscripts a new set of quantities of the same type is obtained by *addition*,

$$P^{ab\ldots c}_{ij\ldots k} + Q^{ab\ldots c}_{ij\ldots k} = S^{ab\ldots c}_{ij\ldots k}.$$

As an illustration of these operations we may write the formula for a multiple power series

$$A + A_i x^i + \frac{1}{2} A_{ij} x^i x^j + \frac{1}{3!} A_{ijk} x^i x^j x^k + \ldots.$$

A set of quantities such as (1·2) is said to be *symmetric* in any set of indices provided that the value of the symbol (1·2) is unaltered by any permutation of the indices in question. For example, if

$$\Gamma^i_{jk} = \Gamma^i_{kj},$$

the quantities Γ are *symmetric* in the subscripts. A set of quantities is said to be *alternating* (or antisymmetric or skew-symmetric) in a given set of indices providing it is unchanged by any even permutation* of the set of indices in question and merely changed in sign by any odd permutation of the same indices. For example, if

$$\Gamma^i_{jk} = -\Gamma^i_{kj},$$

the quantities Γ are alternating in the subscripts.

2. The Kronecker deltas.

The theory of determinants and allied expressions is essentially a theory of alternating sets of quantities, and can be made to depend on certain fundamental alternating sets of quantities which have only the values 0 and + 1 and − 1. These sets of quantities are known as generalized Kronecker deltas because of their analogy with the Kronecker delta which is already well known. The latter is defined as follows:

$$\delta^i_j = 1, \text{ if } i = j; \text{ and } \delta^i_j = 0, \text{ if } i \neq j.$$

Hence using the summation convention

$$(2{\cdot}1) \qquad \delta^i_i = n,$$

and

$$\delta^i_j a_i = a_j, \text{ and } \delta^i_j a^j = a^i.$$

If $x^1, x^2, \ldots, x^n$ are independent variables,

$$(2{\cdot}2) \qquad \frac{\partial x^i}{\partial x^j} = \delta^i_j.$$

The generalized Kronecker delta has k superscripts and k subscripts, each running from 1 to n, and is alternating both in superscripts and subscripts. It is denoted by

$$\delta^{i_1 i_2 \ldots i_k}_{j_1 j_2 \ldots j_k}.$$

If the superscripts are distinct from each other and the subscripts are the same set of numbers as the superscripts, the value of the symbol is + 1 or − 1 according as an even or an odd permutation is required to arrange the superscripts in the same order as the subscripts; in all other cases its value is 0.

* It is proved in books on algebra that any permutation of n objects can be brought about by a finite number of transpositions of pairs of these objects, and that the number of transpositions required to bring about a given permutation is always even or always odd. If this number is even the permutation is said to be even. In the opposite case the permutation is said to be odd.

For example, if $n = 3$ and $k = 2$

$$0 = \delta^{11}_{ij} = \delta^{22}_{ij} = \delta^{ij}_{11} = \delta^{12}_{13}, \text{ etc.} \quad 1 = \delta^{12}_{12} = \delta^{13}_{13} = \delta^{21}_{21}, \text{ etc.}$$

$$-1 = \delta^{12}_{21} = \delta^{13}_{31} = \delta^{21}_{12}, \text{ etc.}$$

Using these symbols, the general formula for any two-rowed determinant formed from the matrix

$$\begin{pmatrix} x^1 x^2 \dots x^n \\ y^1 y^2 \dots y^n \end{pmatrix}$$

is

$$x^i y^j - x^j y^i = \delta^{ij}_{ab}\, x^a y^b.$$

In like manner we can represent differential expressions which are analogous to determinants. For example, if $A_1, A_2, \dots, A_n$ are functions of $x^1, x^2, \dots, x^n$,

$$\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i} = \delta^{ab}_{ij} \frac{\partial A_a}{\partial x^b}.$$

The general three-rowed determinant formed from the matrix

$$\begin{pmatrix} A_1 & A_2 & A_3 & \dots & A_n \\ B_1 & B_2 & B_3 & \dots & B_n \\ C_1 & C_2 & C_3 & \dots & C_n \end{pmatrix}$$

is

$$\delta^{abc}_{ijk}\, A_a B_b C_c = \begin{vmatrix} A_i & A_j & A_k \\ B_i & B_j & B_k \\ C_i & C_j & C_k \end{vmatrix}.$$

If A_i and B_i are analytic functions of $x^1, x^2, \dots, x^n$, we can form determinant-like expressions as follows:

$$\delta^{abc}_{ijk}\, A_a \frac{\partial B_b}{\partial x^c} = A_i \left(\frac{\partial B_j}{\partial x^k} - \frac{\partial B_k}{\partial x^j}\right) + A_j \left(\frac{\partial B_k}{\partial x^i} - \frac{\partial B_i}{\partial x^k}\right) + A_k \left(\frac{\partial B_i}{\partial x^j} - \frac{\partial B_j}{\partial x^i}\right).$$

If a set of quantities T is symmetric in two or more subscripts

$$(2\cdot3) \qquad \delta^{ij\dots l}_{pq\dots r}\, T^{ab\dots c}_{ij\dots l} = 0,$$

and if it is symmétric in two or more superscripts an analogous relation holds. If a set of quantities A is alternating in its subscripts, which are k in number,

$$(2\cdot4) \qquad \delta^{ij\dots l}_{pq\dots r}\, A_{ij\dots l} = k!\, A_{pq\dots r}.$$

3. In studying determinants it is often advantageous to use two other permutation symbols defined as follows:

$$(3\cdot1) \qquad \epsilon^{i_1 i_2 \dots i_n} = \delta^{i_1 i_2 \dots i_n}_{12\dots n} = \delta^{12\dots n}_{i_1 i_2 \dots i_n} = \epsilon_{i_1 i_2 \dots i_n}.$$

Thus ϵ is $+1$ or -1 according as the subscripts or superscripts are obtained from the natural numbers $1\ 2 \ldots n$ by an even or an odd permutation; otherwise it is zero. The number of indices on an epsilon is always n. By the definition of a determinant,

$$(3\cdot2) \qquad a = |\, a^i_j \,| = \begin{vmatrix} a^1_1 & a^1_2 & \ldots & a^1_n \\ a^2_1 & a^2_2 & \ldots & a^2_n \\ \vdots & & & \\ a^n_1 & a^n_2 & \ldots & a^n_n \end{vmatrix}$$

$$= \epsilon^{i_1 i_2 \ldots i_n} a^1_{i_1} a^2_{i_2} \ldots a^n_{i_n} = \epsilon_{i_1 i_2 \ldots i_n} a^{i_1}_1 a^{i_2}_2 \ldots a^{i_n}_n.$$

For example, a generalized Kronecker delta is a determinant of the simple Kronecker deltas,

$$(3\cdot3) \qquad \delta^{i_1 i_2 \ldots i_k}_{j_1 j_2 \ldots j_k} = \epsilon_{ab \ldots c}\, \delta^{i_a}_{j_1} \delta^{i_b}_{j_2} \ldots \delta^{i_c}_{j_k}.$$

(Here we are applying the summation convention to indices and subscripts of indices.) From either of the expansions in (3·2) it is evident that interchanging two rows of the determinant changes its sign. Hence for any permutation of rows

$$(3\cdot41) \qquad a \epsilon^{ab \ldots c} = \epsilon^{ij \ldots k} a^a_i a^b_j \ldots a^c_k.$$

Likewise, for any permutation of columns

$$(3\cdot42) \qquad a \epsilon_{ab \ldots c} = \epsilon_{ij \ldots k} a^i_a a^j_b \ldots a^k_c.$$

It is an obvious corollary of these two formulas that $a = 0$ if any two rows or any two columns of the determinant are identical.

The formula for the product of two determinants may be derived as follows:

$$(3\cdot5) \qquad \begin{aligned} ab &= a\epsilon_{ab \ldots c}\, b^a_1 b^b_2 \ldots b^c_n \\ &= \epsilon_{ij \ldots m}\, a^i_a a^j_b \ldots a^m_c\, b^a_1 b^b_2 \ldots b^c_n \\ &= \epsilon_{ij \ldots m}\, (a^i_a b^a_1)(a^j_b b^b_2) \ldots (a^m_c b^c_n) \\ &= |\, c^i_j \,|, \end{aligned}$$

where

$$(3\cdot6) \qquad c^i_j = a^i_a b^a_j = a^i_1 b^1_j + a^i_2 b^2_j + \ldots + a^i_n b^n_j.$$

The formula for the expansion of a determinant in terms of the elements of the first column and their cofactors may be obtained as follows:

$$(3\cdot7) \qquad a = a^{i_1}_1 \epsilon_{i_1 i_2 \ldots i_n} a^{i_2}_2 \ldots a^{i_n}_n = a^i_1 A^1_i,$$

where

(3·8) $$A_i^1 = \epsilon_{ii_2\ldots i_n} a_2^{i_2} a_3^{i_3} \ldots a_n^{i_n}.$$

More generally,

(3·9) $$a = a_p^j \epsilon_{i_1 i_2 \ldots i_n} a_1^{i_1} \ldots a_{p-1}^{i_{p-1}} \delta_j^{i_p} a_{p+1}^{i_{p+1}} \ldots a_n^{i_n}.$$

Hence if we define the cofactor of a_p^j as

(3·10) $$A_j^p = \epsilon_{i_1 i_2 \ldots i_n} a_1^{i_1} \ldots a_{p-1}^{i_{p-1}} \delta_j^{i_p} a_{p+1}^{i_{p+1}} \ldots a_n^{i_n},$$

we have

(3·11) $$a_q^j A_j^p = a\delta_q^p.$$

This formula gives the expansion of the determinant in terms of the elements of the pth column if $p = q$. In case $p \neq q$ it gives the theorem that the sum of the products of the elements of one column into the cofactors of another is zero. The corresponding theorems about the expansion in terms of the elements of a row are

(3·12) $$a_p^j A_i^p = a\delta_i^j.$$

Although we have spoken about rows and columns it is clear that the visual representation of a determinant may be left to one side when we are using the present notation. The notation takes the place of these other devices. It is not merely an abbreviation; it is a measure for economy of thought. For by arranging that unessential or routine questions are taken care of automatically it enables us to concentrate attention on the new ideas which we have to meet.

4. Linear equations.

To solve a set of linear equations

(4·1) $$\begin{aligned} a_1^1 x^1 + a_2^1 x^2 + \ldots + a_n^1 x^n &= b^1, \\ a_1^2 x^1 + a_2^2 x^2 + \ldots + a_n^2 x^n &= b^2, \\ \vdots \qquad\qquad\qquad \vdots \quad &\quad \vdots \\ a_1^n x^1 + a_2^n x^2 + \ldots + a_n^n x^n &= b^n, \end{aligned}$$

or, as we prefer to write them,

(4·2) $$a_j^i x^j = b^i,$$

we multiply (4·2) by A_i^k and sum with respect to i,

$$a_j^i A_i^k x^j = b^i A_i^k.$$

Using (3·11) this reduces to

$$a\delta_j^k x^j = b^i A_i^k,$$

or, in case $a \neq 0$,

$$x^k = \frac{b^i A^k_i}{a}, \tag{4·3}$$

which is Cramer's rule for the solution of linear equations.

5. Functional determinants.

In our work the determinant which appears most frequently is the Jacobian of n functions of n variables

$$y^i (x^1, x^2, \ldots, x^n),$$

which is defined by the equation

$$\frac{\partial (y^1, y^2, \ldots, y^n)}{\partial (x^1, x^2, \ldots, x^n)} = \left| \frac{\partial y}{\partial x} \right| = \epsilon^{i_1 i_2 \ldots i_n} \frac{\partial y^1}{\partial x^{i_1}} \frac{\partial y^2}{\partial x^{i_2}} \cdots \frac{\partial y^n}{\partial x^{i_n}}. \tag{5·1}$$

For n functions $z^i (y^1, y^2, \ldots, y^n)$ a fundamental theorem on partial differentiation states that

$$\frac{\partial z^i}{\partial x^j} = \frac{\partial z^i}{\partial y^k} \frac{\partial y^k}{\partial x^j}. \tag{5·2}$$

Hence by the theorem on multiplication of determinants (3·6) the functional determinants satisfy the equation

$$\left| \frac{\partial z}{\partial x} \right| = \left| \frac{\partial z}{\partial y} \right| \left| \frac{\partial y}{\partial x} \right|. \tag{5·3}$$

In case the functions z^i are such that

$$z^i (y^1, y^2, \ldots, y^n) = x^i,$$

(5·2) becomes

$$\frac{\partial x^i}{\partial y^k} \frac{\partial y^k}{\partial x^j} = \delta^i_j, \tag{5·4}$$

and (5·3) reduces to

$$\left| \frac{\partial y}{\partial x} \right| = \frac{1}{\left| \frac{\partial x}{\partial y} \right|}. \tag{5·5}$$

For a fixed value of j, (5·4) may be regarded as a set of n linear equations for the determination of n unknowns, $\partial y^k / \partial x^j$, the coefficients being the n^2 quantities $\partial x^i / \partial y^k$. Solving these equations according to § 4, we find

$$\frac{\partial y^i}{\partial x^j} = \frac{\text{cofactor of } \frac{\partial x^j}{\partial y^i} \text{ in } \left| \frac{\partial x}{\partial y} \right|}{\left| \frac{\partial x}{\partial y} \right|} \tag{5·6}$$

Other formulas about functional determinants are:

(5·7) $$\delta^{ab\ldots c}_{ij\ldots k}\left|\frac{\partial y}{\partial x}\right| = \frac{\partial\,(y^a y^b \ldots y^c)}{\partial\,(x^i x^j \ldots x^k)},$$

and

(5·8) $$\delta^{ab\ldots c}_{ij\ldots k} = \frac{\partial\,(x^a x^b \ldots x^c)}{\partial\,(x^i x^j \ldots x^k)}.$$

6. By partial differentiation the formulas (5·4) give rise to the following formulas which are often useful:

(6·1) $$\frac{\partial^2 y^i}{\partial x^a \partial x^b}\frac{\partial x^a}{\partial y^j} + \frac{\partial^2 x^a}{\partial y^j \partial y^k}\frac{\partial y^i}{\partial x^a}\frac{\partial y^k}{\partial x^b} = 0;$$

(6·2) $$\frac{\partial^2 y^i}{\partial x^a \partial x^b}\frac{\partial x^a}{\partial y^j}\frac{\partial x^b}{\partial y^k} + \frac{\partial^2 x^a}{\partial y^j \partial y^k}\frac{\partial y^i}{\partial x^a} = 0;$$

(6·3) $$\frac{\partial^3 y^i}{\partial x^a \partial x^b \partial x^c}\frac{\partial x^a}{\partial y^j} + \frac{\partial^2 y^i}{\partial x^a \partial x^b}\frac{\partial^2 x^a}{\partial y^j \partial y^k}\frac{\partial y^k}{\partial x^c} + \frac{\partial^3 x^a}{\partial y^j \partial y^k \partial y^l}\frac{\partial y^i}{\partial x^a}\frac{\partial y^k}{\partial x^b}\frac{\partial y^l}{\partial x^c}$$
$$+ \frac{\partial^2 x^a}{\partial y^j \partial y^k}\frac{\partial^2 y^i}{\partial x^a \partial x^c}\frac{\partial y^k}{\partial x^b} + \frac{\partial^2 x^a}{\partial y^j \partial y^k}\frac{\partial y^i}{\partial x^a}\frac{\partial^2 y^k}{\partial x^b \partial x^c} = 0;$$

and so on.

7. Derivative of a determinant.

In case the elements of a determinant are functions of $(x^1, x^2, \ldots, x^n)$ we have by differentiating (3·2)

(7·1) $$\frac{\partial a}{\partial x^j} = \epsilon_{i_1 i_2 \ldots i_n}\left(\frac{\partial a^{i_1}_1}{\partial x^j} a^{i_2}_2 \ldots a^{i_n}_n + a^{i_1}_1 \frac{\partial a^{i_2}_2}{\partial x^j} \ldots a^{i_n}_n + \ldots\right)$$
$$= \frac{\partial a^{i_1}_1}{\partial x^j} A^1_{i_1} + \frac{\partial a^{i_2}_2}{\partial x^j} A^2_{i_2} + \ldots$$
$$= \frac{\partial a^b_c}{\partial x^j} A^c_b.$$

In case the determinant in question is the Jacobian of a transformation of coordinates, (5·1), (7·1) reduces to

(7·2) $$\frac{\dfrac{\partial}{\partial x^j}\left|\dfrac{\partial y}{\partial x}\right|}{\left|\dfrac{\partial y}{\partial x}\right|} = \frac{\partial^2 y^b}{\partial x^j \partial x^c}\frac{\partial x^c}{\partial y^b}.$$

8. Numerical relations.

The permutation symbols satisfy a number of numerical relations which are easily verified by counting the number of terms which appear in the various sums. Thus we have

$$\delta^{ij}_{pj} = (n-1)\,\delta^{i}_{p}, \quad \text{and} \quad \delta^{ij}_{ij} = n\,(n-1),$$

(8·1) $$\delta^{i_1 i_2 \ldots i_r i_{r+1} \ldots i_k}_{j_1 j_2 \ldots j_r i_{r+1} \ldots i_k} = \frac{(n-r)!}{(n-k)!}\,\delta^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_r},$$

(8·2) $$\delta^{i_1 i_2 \ldots i_k}_{i_1 i_2 \ldots i_k} = \frac{n!}{(n-k)!},$$

(8·3) $$\epsilon^{ab \ldots m}\,\epsilon_{ab \ldots m} = n!,$$

(8·4) $$\epsilon^{i_1 \ldots i_k i_{k+1} \ldots i_n}\,\epsilon_{j_1 \ldots j_k i_{k+1} \ldots i_n} = (n-k)!\,\delta^{i_1 \ldots i_k}_{j_1 \ldots j_k},$$

(8·5) $$\epsilon^{i_1 \ldots i_k i_{k+1} \ldots i_n}\,\delta^{j_{k+1} \ldots j_n}_{i_{k+1} \ldots i_n} = (n-k)!\,\epsilon^{i_1 \ldots i_k j_{k+1} \ldots j_n},$$

(8·6) $$\delta^{i_1 \ldots i_k i_{k+1} \ldots i_r}_{j_1 \ldots j_k j_{k+1} \ldots j_r}\,\delta^{j_{k+1} \ldots j_r}_{p_{k+1} \ldots p_r} = (r-k)!\,\delta^{i_1 \ldots i_k i_{k+1} \ldots i_r}_{j_1 \ldots j_k p_{k+1} \ldots p_r},$$

(8·7) $$\delta^{i_1 \ldots i_k i_{k+1} \ldots i_r}_{j_1 \ldots j_k j_{k+1} \ldots j_r}\,\delta^{j_{k+1} \ldots j_r}_{i_{k+1} \ldots i_r} = \frac{(n-k)!}{(n-r)!}\,(r-k)!\,\delta^{i_1 \ldots i_k}_{j_1 \ldots j_k},$$

(8·8) $$\epsilon_{i \ldots jk \ldots lm} = (-1)^p\,\epsilon_{i \ldots jmk \ldots l},$$

if p is the number of indices $k \ldots l$.

9. Minors, cofactors, and the Laplace expansion.

The k-rowed *minors* of a determinant a are defined by the formula

(9·1) $$a^{i_1 i_2 \ldots i_k}_{j_1 j_2 \ldots j_k} = \delta^{i_1 i_2 \ldots i_k}_{p_1 p_2 \ldots p_k}\,a^{p_1}_{j_1}\,a^{p_2}_{j_2} \ldots a^{p_k}_{j_k}$$
$$= \delta^{p_1 p_2 \ldots p_k}_{j_1 j_2 \ldots j_k}\,a^{i_1}_{p_1}\,a^{i_2}_{p_2} \ldots a^{i_k}_{p_k}.$$

Thus the one-rowed minors are the elements a^i_j themselves, the two-rowed minors are the determinants,

$$\begin{vmatrix} a^{i_1}_{j_1} & a^{i_2}_{j_1} \\ a^{i_1}_{j_2} & a^{i_2}_{j_2} \end{vmatrix},$$

and so on. The determinant is given by

(9·2) $$a = \frac{1}{n!}\,a^{i_1 i_2 \ldots i_n}_{i_1 i_2 \ldots i_n},$$

and we also have

(9·3) $$a\delta^{i_1 \ldots i_k}_{j_1 \ldots j_k} = \frac{1}{(n-k)!}\,a^{i_1 \ldots i_k i_{k+1} \ldots i_n}_{j_1 \ldots j_k i_{k+1} \ldots i_n}.$$

The *cofactor* of the k-rowed minor (9·1) is the determinant

(9·4) $$A^{j_1 j_2 \ldots j_k}_{i_1 i_2 \ldots i_k} = \frac{1}{(n-k)!}\,\delta^{j_1 j_2 \ldots j_n}_{i_1 i_2 \ldots i_n}\,a^{i_{k+1}}_{j_{k+1}} \ldots a^{i_n}_{j_n}$$
$$= \frac{1}{((n-k)!)^2}\,\delta^{j_1 \ldots j_n}_{i_1 \ldots i_n}\,a^{i_{k+1} \ldots i_n}_{j_{k+1} \ldots j_n},$$

which is equivalent to (3·10) if $k = 1$.

Applying some of the formulas of § 8, we find

$$(9\cdot5)\quad a^{i_1\dots i_k}_{j_1\dots j_k}A^{j_1\dots j_k}_{q_1\dots q_k}=\delta^{p_1\dots p_k}_{j_1\dots j_k}a^{i_1}_{p_1}\dots a^{i_k}_{p_k}\frac{1}{(n-k)!}\delta^{j_1\dots j_k\dots j_n}_{q_1\dots q_k\dots q_n}a^{q_{k+1}}_{j_{k+1}}\dots a^{q_n}_{j_n}$$
$$=\frac{k!}{(n-k)!}\delta^{p_1\dots p_k j_{k+1}\dots j_n}_{q_1\dots q_k q_{k+1}\dots q_n}a^{i_1}_{p_1}\dots a^{i_k}_{p_k}a^{q_{k+1}}_{j_{k+1}}\dots a^{q_n}_{j_n}$$
$$=\frac{k!}{(n-k)!}a^{i_1\dots i_k q_{k+1}\dots q_n}_{q_1\dots q_k q_{k+1}\dots q_n}$$
$$=k!\,a\delta^{i_1\dots i_k}_{q_1\dots q_k},$$

a formula which includes the Laplace expansion, and therefore, also (3·11), as a special case. Similarly we can get

$$(9\cdot6)\qquad \delta^{j_1\dots j_n}_{s_1\dots s_n}a^{i_1\dots i_k}_{j_1\dots j_k}a^{i_{k+1}\dots i_n}_{j_{k+1}\dots j_n}=k!\,(n-k)!\,a^{i_1\dots i_n}_{s_1\dots s_n},$$

which also includes the Laplace expansion.

10. For reference later on it will be convenient to write out the formulas for the minors and cofactors of a matrix

$$(10\cdot1)\qquad ||g_{ij}||=\begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1n}\\ g_{21} & g_{22} & \cdots & g_{2n}\\ \vdots & \vdots & & \vdots\\ g_{n1} & g_{n2} & \cdots & g_{nn}\end{pmatrix},$$

in a notation which is slightly different in appearance from that just given, though in fact equivalent to it. The k-rowed minors are

$$(10\cdot2)\qquad g_{ab\dots c;\,ij\dots m}=\delta^{pq\dots r}_{ab\dots c}g_{pi}g_{qj}\cdots g_{rm}.$$

According to this definition the one-rowed minors are the elements g_{ij} themselves. The two-rowed minors satisfy the conditions

$$(10\cdot3)\qquad g_{ab;\,ij}=-g_{ba;\,ij}$$
$$=-g_{ab;\,ji},$$

and the k-rowed minors satisfy analogous relations. In the important special case in which

$$(10\cdot4)\qquad g_{ij}=g_{ji},$$

we also have

$$(10\cdot5)\qquad g_{ab;\,ij}=g_{ij;\,ab}.$$

The cofactors of order k are given by the formulas

$$(10\cdot6)\quad G^{a_1\dots a_k;\,i_1\dots i_k}=\frac{1}{(n-k)!}\epsilon^{a_1\dots a_n}\epsilon^{i_1\dots i_n}g_{a_{k+1}i_{k+1}}\cdots g_{a_n i_n}.$$