THE LEBESGUE INTEGRAL

CHAPTER I

SETS OF POINTS

The refinements of the differential and integral calculus, which form the topic of this tract, largely depend on the properties of *sets of points* in one or more dimensions. This chapter contains those properties that will be needed, in so far as they are *descriptive* and not *metrical*. The rules of algebra applied to *sets* hold whether the members of the sets are points or are objects or concepts of any kind. All that we require for a set E to be defined is that we can say of any given object x whether it is or is not a member of E.

1·1. The algebra of sets. Let E be a set,† the members of which may be of any nature. The *sum* of two sets E_1, E_2 is defined to be the set of objects which belong either to E_1 or to E_2 (or to both); the sum is written E_1+E_2. By definition E_2+E_1 is the same as E_1+E_2, no question of order being involved. The definition extends to any finite or infinite number of sets, $E_1+E_2+\ldots$ being the set of objects belonging to at least one E_n. In the definition of an infinite sum there is no appeal to any limiting process.

The *product* $E_1E_2\ldots$ of any number (finite or infinite) of sets $E_1, E_2, \ldots$ is defined to be the set of objects belonging to every one of the sets E_n. The sets E_n may have no members common to all of them, and the product is then the null set—the set which has no members.

† *Class* and *aggregate* are synonymous with *set*; French *ensemble*, German *Menge*.

If every member of E_1 is a member of E_0 we say that E_1 is contained in E_0 and we write $E_1 \subset E_0$ (or $E_0 \supset E_1$). The set of members of E_0 which do not belong to E_1, may be written $E_0 - E_1$ or alternatively as CE_1, the *complement* of E_1. It is easy to see that, the complements being taken with respect to a fixed E_0,

$$C(E_1 + E_2 + \dots) = CE_1 . CE_2 \dots,$$

and

$$C(E_1 E_2 \dots) = CE_1 + CE_2 + \dots.$$

Limit sets. If $E_1, E_2, \dots$ is an infinite sequence of sets, the upper limit, $\overline{\lim}\, E_n$, is defined to be the set of objects which belong to infinitely many of the E_n. The lower limit $\underline{\lim}\, E_n$ is defined to be the set of objects, each of which belongs to all but a finite number of the E_n. Clearly $\overline{\lim}\, E_n \supset \underline{\lim}\, E_n$. If the sets $\overline{\lim}\, E_n$, $\underline{\lim}\, E_n$ are the same, we say that the sequence $E_1, E_2, \dots$ has a limit, $\lim E_n$.

If a sequence of sets is increasing or decreasing, it has a limit: more precisely,

(*a*) *If* $E_n \subset E_{n+1}$, *then* $\lim E_n = E_1 + E_2 + \dots,$

(*b*) *If* $E_n \supset E_{n+1}$, *then* $\lim E_n = E_1 E_2 \dots.$

To prove (*a*), write $E = E_1 + E_2 + \dots$ and observe that if x is a member of $\overline{\lim}\, E_n$ then x is a member of E; hence $\overline{\lim}\, E_n \subset E$. The result will now follow if we prove that $E \subset \underline{\lim}\, E_n$. This is true because any member of E is a member of E_n for some n and so (since the sets form an increasing sequence) for all greater n and therefore of $\underline{\lim}\, E_n$.

A similar proof holds for (*b*), the product set possibly being null.

More generally, the upper and lower limits of any sequence of sets, not necessarily monotonic (increasing or decreasing), may be expressed in terms of sums and products. The formulae are

$$\overline{\lim}\, E_n = (E_1 + E_2 + E_3 + \dots)(E_2 + E_3 + \dots)(E_3 + \dots)\dots,$$
$$\underline{\lim}\, E_n = E_1 E_2 E_3 \dots + E_2 E_3 \dots + E_3 \dots + \dots.$$

The proof is left to the reader.

1·2. Infinite sets. Two sets are called *similar* if there is a one-one correspondence between the members of one and the members of the other. Thus two sets with finitely many members are similar if and only if each has the same number of members. The idea of similarity is the foundation of any theory of infinite numbers. We shall give here only those outlines of this topic which are essential for later chapters.

With infinite sets we have a phenomenon which cannot occur with finite sets, namely, that a set can be similar to a part of itself. For instance, the set of positive integers is similar to the set of even integers or to the set of perfect cubes.

Any set which is similar to the set of all positive integers or to a finite sub-set of them is said to be *enumerable*. The one-one correspondence may be displayed by using the positive integers as suffixes, so that the members of any enumerable set may be specified as $x_1, x_2, x_3, \ldots$.

It is clear that any sub-set of an enumerable set is enumerable.

The members of an enumerable set of enumerable sets $E_1, E_2, \ldots$ *form an enumerable set.*

For let the members of E_m be enumerated as $x_{m1}, x_{m2}, x_{m3}, \ldots$ The members of all the sets then form a double array:

$$\begin{matrix} x_{11} & x_{12} & x_{13} & \cdots \\ x_{21} & x_{22} & x_{23} & \cdots \\ x_{31} & x_{32} & x_{33} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{matrix}$$

This array can be enumerated as a single sequence, for example, by taking terms along the successive diagonals in order

$$x_{11}, x_{12}, x_{21}, x_{13}, x_{22}, x_{31}, \ldots.$$

As a particular case of this, the set of positive rational numbers is enumerable, for they are all included in the set

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{2}{2}, \frac{3}{1}, \ldots.$$

Clearly, the set of all rational numbers (positive, negative, or zero) is enumerable.

A further application of the same argument proves that *the points of the plane of which both co-ordinates are rational form an enumerable set.*

For if $r_1, r_2, \ldots$ are the rationals enumerated, the rational points of the plane can be displayed as

$$\begin{matrix} (r_1, r_1) & (r_1, r_2) & (r_1, r_3) & \ldots \\ (r_2, r_1) & (r_2, r_2) & (r_2, r_3) & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{matrix}$$

and form an enumerable set of enumerable sets.

The simplest example of a set which is *not* enumerable is the set of all points of an interval. Take the interval (0, 1) and suppose, on the contrary, that all the numbers between 0 and 1 can be enumerated as $x_1, x_2, x_3, \ldots$. Let each x_n be expressed as a decimal

$$x_n = \cdot u_{n1} u_{n2} u_{n3} \ldots,$$

the u's being numbers from 0 to 9. Write down a new number,

$$y = \cdot v_1 v_2 v_3 \ldots,$$

where v_n is determined from u_{nn} by the rule that $v_n = 1$ if $u_{nn} \neq 1$ and $v_n = 2$ if $u_{nn} = 1$. Then y lies between 0 and 1 and is not the same as any x_n, for it differs from x_n in the nth decimal place. This contradicts the hypothesis that the sequence $x_1, x_2, \ldots$ included all the numbers of (0, 1).

1·3. Sets of points. Descriptive properties. Suppose that E is a set of points on a line.

E is *bounded* if all its points are included in some finite interval.

A point P, of abscissa x, is said to be an *interior point* of E if there is a neighbourhood $(x-\delta, x+\delta)$ of P, every point of which belongs to E.

A set E is said to be *open* if every point of it is an interior point. The simplest open set is an interval $a<x<b$ without its end-points.

A point P, of abscissa x (which may or may not be a point of E) is said to be a *limit-point* of E if any neighbourhood $(x-\delta, x+\delta)$, however small δ, contains a point of E other than P. It follows that every neighbourhood of P contains infinitely many points of E. The set of all limit-points of E is called the *derived set* of E and is denoted by E'.

Weierstrass proved that, if E is a bounded set having infinitely many points, then E' contains at least one point.†

A set E is said to be *closed* if $E \supset E'$. (For example, consider the closed interval $a \leqslant x \leqslant b$.)

It will be convenient to reserve the letters O and Q, with suffixes, for open and closed sets respectively. We shall prove first that these two ideas are complementary.

If Q is closed, then CQ is open (the complement naturally must be taken with respect to an *open* interval).

For let P be a point of CQ. Since Q contains all its limit-points, P is not a limit-point of Q. Therefore there is a neighbourhood of P free of points of Q, and so CQ is open.

If O is open, then CO (taken with respect to a closed interval) is closed.

For no point of O is a limit-point of points of CO.

It is to be observed that the set of all points of the line $(-\infty<x<\infty)$ is both open and closed, and so is the null set (the set which contains no points).

If E is any set, E' is closed.

If E' has a limit-point P, let I be a neighbourhood of P. I contains a point of E', say P_1. Let I_1 be a neighbourhood of P_1 contained in I. Then infinitely many points of E lie in I_1 and so in I. Hence P is a limit-point of E and therefore is a point of E'.

The set $\overline{E}=E+E'$ is called the *closure* of E.

† Hardy, *A Course of Pure Mathematics*, ch. I.

A linear open set is the sum of an enumerable set of open intervals.

Every point of O is contained in an interval I, consisting entirely of points of O, whose end-points belong to CO. No two of the intervals I overlap. To prove that they form an enumerable set, suppose that $P_1, P_2, \ldots$ are the rational points of the line, enumerated. With each interval I associate the P_n of smallest suffix contained in it. We thus have a one-one correspondence between the intervals I and a sub-class of the positive integers, and the theorem is proved.

The sum of (a finite number or) an infinity of open sets is open.

For if P is a point of ΣO_n it is a point of some O_n. It is an interior point of O_n and *a fortiori* an interior point of ΣO_n.

The result complementary to the last is that the product of infinitely many closed sets is closed. The product set may be null. There is one important case in which we can assert that it is not null, namely, that of a decreasing sequence of bounded sets.

If $Q_n \subset (a, b)$ *and* $Q_n \supset Q_{n+1}$, *then* ΠQ_n *is closed and not null.*

Let P_n be the left-hand end-point of Q_n. The P_n have a limit-point P. P is contained in every Q_n and therefore in ΠQ_n.

1·4. Covering theorems. In proving theorems about real functions, we often have the situation that every point of a set E is associated by some property or other with an interval containing it. In general these intervals form a non-enumerable set and we desire to select an enumerable (or finite) sub-set which cover every point of E. The reader may well have first met this problem in a discussion of the properties of a continuous function $f(x)$ in an interval $a \leqslant x \leqslant b$. If, for a given ϵ, we associate with a value x the greatest interval within which the function lies between $f(x) \pm \epsilon$, then it can be shown that a finite number of these intervals will serve to 'cover' (a, b), and this yields the property of uniform continuity.†

We prove two covering theorems.

† Hardy, *Pure Mathematics*, pp. 196–201.

LINDELÖF'S THEOREM. *To each point x of a set E corresponds an open interval $I(x)$ containing x. Then there is an enumerable set of these intervals covering E.*

The rational intervals of the line (i.e. intervals with rational end-points) form an enumerable set. An interval $I(x)$ containing x includes a rational interval containing x. Thus E is covered by an enumerable set of rational intervals, and *a fortiori* by an enumerable set of the $I(x)$.

THE HEINE-BOREL THEOREM. *If the E of Lindelöf's theorem is bounded and closed, it can be covered by a finite number of the associated $I(x)$.*

Let $I_1, I_2, \ldots$ be a Lindelöf covering of E. Let E_n be the part of E outside $I_1 + I_2 + \ldots + I_n$. Then E_n is closed. The theorem follows if, for a sufficiently large n, E_n is null. Suppose that no E_n is null. Then, since the E_n form a decreasing sequence of bounded closed sets, there is a point common to all of them. This point is in E and is in no I_n, and we have a contradiction.

1·5. Plane sets. The reader will satisfy himself that many of the properties which we have for simplicity established for linear sets are true of sets in higher dimensions. Some care is necessary. The natural plane analogue of a linear interval is a rectangle. It is not true that the general open set in the plane can be decomposed into non-overlapping open rectangles in the same way that a linear open set is the sum of intervals. The standard decomposition of a plane open set is into rectangles, not overlapping but adjoining, i.e. having sides and parts of sides in common. For carrying out this process in a systematic way, and for other purposes, the idea of a *network* is useful.

Take a pair of axes and the lines $x = \pm n$, $y = \pm n$ for all integral values. Denote by G_1 the set of all squares of side 1 so formed. Each square is to be regarded as closed, i.e. its sides belong to it. The lines $x = \pm \frac{1}{2}n$, $y = \pm \frac{1}{2}n$ will now form a set of squares of side $\frac{1}{2}$; call this set G_2. Continue this process of bisection; for every integral m we have a set of squares G_m each of side $1/2^{m-1}$. Call this construction a network and any square appearing in it a *mesh*. The set of meshes is enumerable.

Any point of the plane is then determined by a sequence of meshes $g_1 \supset g_2 \supset \ldots$, where g_m is a mesh of G_m.

We apply this construction to prove the theorem about plane open sets.

A plane open set is the sum of an enumerable set of closed rectangles.

Let O be the set, and P any point of it. Since O is open, a mesh g_m containing P will, if m is sufficiently large, be wholly contained in O. Then O consists of the meshes of G_1 contained in it, the meshes of G_2 contained in O but not in G_1, the meshes of G_3 contained in O but not in G_1 or G_2, and so on. This proves the theorem.

These remarks about sets and networks in a plane can be extended to higher dimensions.

EXAMPLES ON CHAPTER I

(1) If E_n and F_n are two sequences of sets, prove that

$$\underline{\lim}\, E_n + \underline{\lim}\, F_n \subset \underline{\lim}\,(E_n + F_n) \subset \underline{\lim}\, E_n + \overline{\lim}\, F_n \subset \overline{\lim}\,(E_n + F_n) = \overline{\lim}\, E_n + \overline{\lim}\, F_n.$$

Establish a similar chain of inequalities for products. Deduce that, if $\lim E_n$ and $\lim F_n$ exist, then so do $\lim (E_n + F_n)$ and $\lim (E_n F_n)$.

(2) If, for any choice of a finite number of values $x_1, x_2, \ldots, x_n$, the sum $\sum_{r=1}^{n} f(x_r)$ is bounded, prove that the set of values of x for which $f(x) \neq 0$ is enumerable.

Deduce that the values of x for which a given increasing function is discontinuous form an enumerable set.

(3) Let E be the set of points

$$x = \frac{1}{2^l} + \frac{1}{3^m} + \frac{1}{5^n},$$

where l, m, n have all positive integral values. What are the first, second and third derived sets E', E'', E'''?

(4) If $f(x)$ is a continuous function, and A is any constant, prove that the set $E(f \geq A)$ of values of x for which $f(x) \geq A$ is closed.

Prove that the same result is true under the more general hypothesis that $f(x)$ is *upper semi-continuous*, i.e. for each ξ,

$$\overline{\lim_{x \to \xi}} f(x) \leq f(\xi).$$

(5) A point of E which is not a limit-point is called *isolated*. A set all of whose points are isolated is called an isolated set. Prove that an isolated set is enumerable.

(6) Prove that the set of maxima of a given function $f(x)$ is enumerable.

(7) A set E for which $E = E'$ is called *perfect*. (The simplest perfect set is a closed interval.) Prove that the following construction gives a perfect set (Cantor's ternary set).

From the closed interval $(0, 1)$ remove the middle third, the interval $(\frac{1}{3}, \frac{2}{3})$, taken as open. Remove the (open) middle thirds of each of the two remaining intervals $(0, \frac{1}{3})$ and $(\frac{2}{3}, 1)$. This will leave four intervals. Remove the middle third of each of them. Continue the process indefinitely. The set of points which remain is perfect.

(8) Prove that the perfect set of Ex. 7 is not enumerable.

CHAPTER II

MEASURE

2·1. Measure. Following Borel and Lebesgue we shall aim at assigning to a set of points on a line a number called its *measure* which shall generalize the idea of length. The measure of an interval will be its length. For the theory to be satisfactory we shall want, for example, the measure of the sum of two sets without common points to be equal to the sum of the measures of the sets, that is to say, measure is to be an *additive function* of sets of points.

If O is an open set, the additive property requires us to define its measure mO to be the sum of the lengths of its constituent intervals; it is assumed that this sum is convergent (it will certainly be convergent if O is contained in a finite interval).

2·2. Measure of open sets. Let O_1 and O_2 be open sets with no common points. Since the measure of an open set is the sum of the lengths of its intervals we have

$$m(O_1+O_2) = mO_1+mO_2.$$

More generally, we shall prove that, if O_1 and O_2 are any two open sets, then

$$m(O_1+O_2)+m(O_1O_2) = mO_1+mO_2.$$

Taking first a special case, if O_1 and O_2 consist of a finite number (n) of intervals, we can prove the result by induction on n.

Generally, let O_1 and O_2 be any open sets; we can suppose their intervals enumerated (p. 6). Let $\epsilon_\nu \to 0$ as $\nu\to\infty$. Take n such that I_n and J_n, the sums of the first n intervals of O_1 and O_2 respectively, satisfy

$$mO_1 - mI_n < \epsilon_\nu \quad \text{and} \quad mO_2 - mJ_n < \epsilon_\nu.$$