Chapter 1

Classical Banach Spaces

To begin, recall that a Banach space is a complete normed linear space. That is, a Banach space is a normed vector space $(X, \|\cdot\|)$ that is a complete metric space under the induced metric $d(x, y) = \|x - y\|$. Unless otherwise specified, we'll assume that all vector spaces are over $\mathbb{R}$, although, from time to time, we will have occasion to consider vector spaces over $\mathbb{C}$.

What follows is a list of the *classical* Banach spaces. Roughly translated, this means the spaces known to Banach. Once we have these examples out in the open, we'll have plenty of time to fill in any unexplained terminology. For now, just let the words wash over you.

The Sequence Spaces ℓ_p and c_0

Arguably the first infinite-dimensional Banach spaces to be studied were the sequence spaces ℓ_p and c_0. To consolidate notation, we first define the vector space s of all real sequences $x = (x_n)$ and then define various subspaces of s.

For each $1 \le p < \infty$, we define

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}$$

and take ℓ_p to be the collection of those $x \in s$ for which $\|x\|_p < \infty$. The inequalities of Hölder and Minkowski show that ℓ_p is a normed space; from there it's not hard to see that ℓ_p is actually a Banach space.

The space ℓ_p is defined in exactly the same way for $0 < p < 1$ but, in this case, $\|\cdot\|_p$ defines a complete quasi-norm. That is, the triangle inequality now holds with an extra constant; specifically, $\|x + y\|_p \le 2^{1/p}(\|x\|_p + \|y\|_y)$. It's worth noting that $d(x, y) = \|x - y\|_p^p$ defines a complete, translation-invariant metric on ℓ_p for $0 < p < 1$.

For $p = \infty$, we define ℓ_∞ to be the collection of all bounded sequences; that is, ℓ_∞ consists of those $x \in s$ for which

$$\|x\|_\infty = \sup_n |x_n| < \infty.$$

It's easy to see that convergence in ℓ_∞ is the same as uniform convergence on $\mathbb{N}$ and, hence, that ℓ_∞ is complete. There are two very natural (closed) subspaces of ℓ_∞: The space c, consisting of all convergent sequences, and the space c_0, consisting of all sequences converging to 0. It's not hard to see that c and c_0 are also Banach spaces.

As subsets of s we have

$$\ell_1 \subset \ell_p \subset \ell_q \subset c_0 \subset c \subset \ell_\infty \tag{1.1}$$

for any $1 < p < q < \infty$. Moreover, each of the inclusions is norm one:

$$\|x\|_1 \geq \|x\|_p \geq \|x\|_q \geq \|x\|_\infty. \tag{1.2}$$

It's of some interest here to point out that, although s is not itself a normed space, it is, at least, a complete metric space under the so-called Fréchet metric

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}. \tag{1.3}$$

Clearly, convergence in the Fréchet metric implies coordinatewise convergence.

Finite-Dimensional Spaces

We will also have occasion to consider the finite-dimensional versions of the ℓ_p spaces. We write ℓ_p^n to denote $\mathbb{R}^n$ under the ℓ_p norm. That is, ℓ_p^n is the space of all sequences $x = (x_1, \ldots, x_n)$ of length n and is supplied with the norm

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

for $p < \infty$, and

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$$

for $p = \infty$.

Recall that all norms on $\mathbb{R}^n$ are equivalent. In particular, given any norm $\|\cdot\|$ on $\mathbb{R}^n$, we can find a positive, finite constant C such that

$$C^{-1}\|x\|_1 \leq \|x\| \leq C\|x\|_1 \tag{1.4}$$

for all $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$. Thus, convergence in any norm on $\mathbb{R}^n$ is the same as "coordinatewise" convergence and, hence, every norm on $\mathbb{R}^n$ is complete.

Because every finite-dimensional normed space is just "$\mathbb{R}^n$ in disguise," it follows that every finite-dimensional normed space is complete.

The L_p Spaces

We first define the vector space $L_0[0, 1]$ to be the collection of all (equivalence classes, under equality almost everywhere [a.e.], of) Lebesgue-measurable functions $f : [0, 1] \to \mathbb{R}$. For our purposes, L_0 will serve as the "measurable analogue" of the sequence space s.

For $1 \le p < \infty$, the Banach space $L_p[0, 1]$ consists of those $f \in L_0[0, 1]$ for which

$$\|f\|_p = \left(\int_0^1 |f(x)|^p \, dx \right)^{1/p} < \infty.$$

The space $L_\infty[0, 1]$ consists of all (essentially) bounded $f \in L_0[0, 1]$ under the essential supremum norm

$$\|f\|_\infty = \operatorname*{ess.sup}_{0 \le x \le 1} |f(x)| = \inf\{B : |f| \le B \text{ a.e.}\}$$

(in practice, though, we often just write "sup" in place of "ess.sup"). Again, the inequalities of Hölder and Minkowski play an important role here.

As before, the spaces $L_p[0, 1]$ are also defined for $0 < p < 1$, but $\|\cdot\|_p$ defines only a quasi-norm. Again, $d(f, g) = \|f - g\|_p^p$ defines a complete, translation-invariant metric on L_p for $0 < p < 1$. The space $L_0[0, 1]$ is given the topology of convergence (locally) in measure. For Lebesgue measure on $[0, 1]$, this topology is known to be equivalent to that given by the metric

$$d(f, g) = \int_0^1 \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|} \, dx. \tag{1.5}$$

As subsets of $L_0[0, 1]$, we have the following inclusions:

$$L_1[0, 1] \supset L_p[0, 1] \supset L_q[0, 1] \supset L_\infty[0, 1], \tag{1.6}$$

for any $1 < p < q < \infty$. Moreover, the inclusion maps are all norm one:

$$\|f\|_1 \le \|f\|_p \le \|f\|_q \le \|f\|_\infty. \tag{1.7}$$

The spaces $L_p(\mathbb{R})$ are defined in much the same way but satisfy *no* such inclusion relations. That is, for any $p \neq q$, we have $L_p(\mathbb{R}) \not\subset L_q(\mathbb{R})$. Nevertheless, you may find it curious to learn that $L_p(\mathbb{R})$ and $L_p[0, 1]$ are linearly *isometric*.

More generally, given a measure space (X, Σ, μ), we might consider the space $L_p(\mu)$ consisting of all (equivalence classes of) Σ-measurable functions $f : X \to \mathbb{R}$ under the norm

$$\|f\|_p = \left(\int_X |f(x)|^p \, d\mu(x) \right)^{1/p}$$

(with the obvious modification for $p = \infty$).

It is convenient to consider at least one special case here: Given any set Γ, we define $\ell_p(\Gamma) = L_p(\Gamma, 2^\Gamma, \mu)$, where μ is counting measure on Γ. What this means is that we identify functions $f : \Gamma \to \mathbb{R}$ with "sequences" $x = (x_\gamma)$ in the usual way: $x_\gamma = f(\gamma)$, and we define

$$\|x\|_p = \left(\sum_{\gamma \in \Gamma} |x_\gamma|^p \right)^{1/p} = \left(\int_\Gamma |f(\gamma)|^p \, d\mu(\gamma) \right)^{1/p} = \|f\|_p$$

for $p < \infty$. Please note that if $x \in \ell_p(\Gamma)$, then $x_\gamma = 0$ for all but countably many γ. For $p = \infty$, we set

$$\|x\|_\infty = \sup_{\gamma \in \Gamma} |x_\gamma| = \sup_{\gamma \in \Gamma} |f(\gamma)| = \|f\|_\infty.$$

We also define $c_0(\Gamma)$ to be the space of all those $x \in \ell_\infty(\Gamma)$ for which the set $\{\gamma : |x_\gamma| > \varepsilon\}$ is *finite* for any $\varepsilon > 0$. Again, this forces an element of $c_0(\Gamma)$ to have countable support. Clearly, $\ell_p(\mathbb{N}) = \ell_p$ and $c_0(\mathbb{N}) = c_0$.

A priori, the Banach space characteristics of $L_p(\mu)$ will depend on the underlying measure space (X, Σ, μ). As it happens, though, Lebesgue measure on $[0, 1]$ and counting measure on $\mathbb{N}$ are essentially the only two cases we have to worry about. It follows from a deep result in abstract measure theory (Maharam's theorem [97]) that every complete measure space can be decomposed into "nonatomic" parts (copies of $[0, 1]$) and "purely atomic" parts (counting measure on some discrete space). From a Banach space point of view, this means that every L_p space can be written as a direct sum of copies of $L_p[0, 1]$ and $\ell_p(\Gamma)$ (or ℓ_p^n).

For the most part we will divide our efforts here into three avenues of attack: Those properties of L_p spaces that don't depend on the underlying measure space, those that are peculiar to $L_p[0, 1]$, and those that are peculiar to the ℓ_p spaces.

The $C(K)$ Spaces

Perhaps the earliest known example of a Banach space is the space $C[a, b]$ of all continuous real-valued functions $f : [a, b] \to \mathbb{R}$ supplied with the "uniform norm":

$$\|f\| = \max_{a \le t \le b} |f(t)|.$$

More generally, if K is any compact Hausdorff space, we write $C(K)$ to denote the Banach space of all continuous real-valued functions $f : K \to \mathbb{R}$ under the norm

$$\|f\| = \max_{t \in K} |f(t)|.$$

For obvious reasons, we sometimes write the norm in $C(K)$ as $\|f\|_\infty$ and refer to it as the "sup norm." In any case, convergence in $C(K)$ is the same as uniform convergence on K.

In Banach's day, point set topology was still very much in its developmental stages. In his book [6], Banach considered $C(K)$ spaces only in the case of compact *metric* spaces K. We, on the other hand, may have occasion to venture further. At the very least, we will consider the case in which K is a compact Hausdorff space (since the theory is nearly identical in this case). And, if we really get ambitious, we may delve into more esoteric settings. For the sake of future reference, here is a brief summary of the situation.

If X is any topological space, we write $C(X)$ to denote the algebra of all real-valued continuous functions $f : X \to \mathbb{R}$. For general X, though, $C(X)$ may not be metrizable. If X is Hausdorff and σ-compact, say $X = \bigcup_{n=1}^\infty K_n$, then $C(X)$ is a complete metric space under the topology of "uniform convergence on compacta" (or the "compact-open" topology). This topology is generated by the so-called Fréchet metric

$$d(f, g) = \sum_{n=1}^\infty 2^{-n} \frac{\|f - g\|_n}{1 + \|f - g\|_n}, \tag{1.8}$$

where $\|f\|_n$ is the norm of $f|_{K_n}$ in $C(K_n)$.

If we restrict our attention to the *bounded* functions in $C(X)$, then we may at least apply the sup norm; for this reason, we consider instead the Banach space $C_b(X)$ of all bounded, continuous, real-valued functions $f : X \to \mathbb{R}$ endowed with the sup norm

$$\|f\| = \sup_{x \in X} |f(x)|.$$

Obviously, $C_b(X)$ is a closed subspace of $\ell_\infty(X)$. If X is at least completely regular, then $C_b(X)$ contains as much information as $C(X)$ itself in the sense that the topology on X is completely determined by knowing the bounded, continuous, real-valued functions on X.

If X is noncompact, then we might also consider the normed space $C_C(X)$ of all continuous $f : X \to \mathbb{R}$ with *compact support*. That is, $f \in C_C(X)$ if f is continuous and if the *support* of f, namely, the set

$$\text{supp } f = \overline{\{x \in X : f(x) \neq 0\}},$$

is compact. Although we may apply the sup norm to $C_C(X)$, it's not, in general, complete. The completion of $C_C(X)$ is the space $C_0(X)$ consisting of all those continuous $f : X \to \mathbb{R}$ that "vanish at infinity." Specifically, $f \in C_0(X)$ if f is continuous and if, for each $\varepsilon > 0$, the set $\{|f| \geq \varepsilon\}$ has compact closure. The space $C_0(X)$ is a closed subspace of $C_b(X)$ and hence is a Banach space under the sup norm.

If X is compact, then, of course, $C_C(X) = C_b(X) = C(X)$. For general X, however, the best we can say is

$$C_C(X) \subset C_0(X) \subset C_b(X) \subset C(X).$$

At least one easy example might be enlightening here: Consider the case $X = \mathbb{N}$; obviously, $\mathbb{N}$ is locally compact and metrizable. Now *every* function $f : \mathbb{N} \to \mathbb{R}$ is continuous, and any such function can quite plainly be identified with a sequence; namely, its range $(f(n))$. That is, we can identify $C(\mathbb{N})$ with s by way of the correspondence $f \in C(\mathbb{N}) \longleftrightarrow x \in s$, where $x_n = f(n)$. Convince yourself that

$$C_b(\mathbb{N}) = \ell_\infty, \quad C_0(\mathbb{N}) = c_0, \quad C_0(\mathbb{N}) \oplus \mathbb{R} = c, \tag{1.9}$$

and that

$$C_C(\mathbb{N}) = \{x \in s : x_n = 0 \text{ for all but finitely many } n\}. \tag{1.10}$$

While this is curious, it doesn't quite tell the whole story. Indeed, both ℓ_∞ and c are actually $C(K)$ spaces. To get a glimpse into why this is true, consider the space $\mathbb{N}^* = \mathbb{N} \cup \{\infty\}$, the *one-point compactification* of $\mathbb{N}$ (that is, we append a "point at infinity"). If we define a neighborhood of ∞ to be any set with finite (compact) complement, then $\mathbb{N}^*$ becomes a compact Hausdorff space. Convince yourself that

$$c = C(\mathbb{N}^*) \qquad \text{and} \qquad c_0 = \{f \in C(\mathbb{N}^*) : f(\infty) = 0)\}. \tag{1.11}$$

We'll have more to say about these ideas later.

Hilbert Space

As you'll no doubt recall, the spaces ℓ_2 and L_2 are both Hilbert spaces, or complete inner product spaces. Recall that a vector space H is called a Hilbert space if H is endowed with an inner product $\langle \cdot, \cdot \rangle$ with the property that the induced norm, defined by

$$\|x\| = \sqrt{\langle x, x \rangle}, \tag{1.12}$$

is complete. It is most important here to recognize that the norm in H is intimately related to an inner product by way of (1.12). This is a tall order for the run-of-the-mill norm. From this point of view, Hilbert spaces are quite rare among the teeming masses of Banach spaces.

There is a critical distinction to be made here; perhaps an example will help to explain. Let X denote the space ℓ_2 supplied with the norm $\|x\| = \|x\|_2 + \|x\|_\infty$. Then X is isomorphic (linearly homeomorphic) to ℓ_2 because our new norm satisfies $\|x\|_2 \le \|x\| \le 2\|x\|_2$. But X is *not* itself a Hilbert space. The test is whether the *parallelogram law* holds:

$$\|x + y\|^2 + \|x - y\|^2 \stackrel{?}{=} 2\left(\|x\|^2 + \|y\|^2\right).$$

And it's easy to check that the parallelogram law fails if $x = (1, 0, 0, \ldots)$ and $y = (0, 1, 0, \ldots)$, for instance. The moral here is that it's not enough to have a well-defined inner product, nor is it enough to have a norm that is close to a known Hilbert space norm. In a Hilbert space, the norm and the inner product are inextricably bound together through equation (1.12).

Hilbert spaces exhibit another property that is rare among the Banach spaces: In a Hilbert space, every closed subspace is the range of a continuous projection. This is far from the case in a general Banach space. (In fact, it is known that any space with this property is already isomorphic to Hilbert space.)

"Neoclassical" Spaces

We have more or less exhausted the list of spaces that were well known in Banach's time. But we have by no means even begun to list the spaces that are commonplace these days. In fact, it would take pages and pages of definitions to do so. For now we'll content ourselves with the understanding that all of the known examples are, in a sense, generalizations of the spaces we have seen thus far.

The Big Questions

We're typically interested in both the *isometric* as well as the *isomorphic* character of a Banach space. (For our purposes, all isometries are *linear*.

Also, as a reminder, an isomorphism is a *linear* homeomorphism.) Here are just a few of the questions we might consider:

- ▹ Are all the spaces listed above isometrically distinct? For example, is it at all possible that ℓ_4 and ℓ_6 are isometric? What about ℓ_p and L_p? Or $L_p[0, 1]$ and $L_p(\mathbb{R})$?
- ▹ When is a given Banach space X isometric to a subspace of one of the classical spaces? When does X contain an isometric copy of one of the classical spaces? In particular, does L_1 embed isometrically into L_2? Does ℓ_p embed isometrically into $C[0, 1]$?
- ▹ We might pose all of the preceding questions, replacing the word "isometric" with "isomorphic."
- ▹ Characterize all of the subspaces of a given Banach space X, if possible, both isometrically and isomorphically. In particular, identify those subspaces that are the range of a continuous projection (that is, the *complemented* subspaces of X). Knowing all of the subspaces of a given space would tell us something about the linear operators into or on the space. (And vice versa: After all, the kernel and range of a linear operator are subspaces.)
- ▹ All of the spaces we've defined above carry some additional structure. $C[a, b]$ is an algebra of functions, for example, and $L_1[0, 1]$ is a lattice. What, if anything, does this extra structure tell us from the point of view of Banach space theory? Is it an isometric invariant of these spaces? An isomorphic invariant? Does it imply the existence of interesting subspaces? Or interesting operators?
- ▹ It's probably fair to say that functional analysis concerns the study of *operators* between spaces. Insert the adjective "linear," wherever possible, and you will have a working definition of *linear* functional analysis. Where does the study of Banach spaces fit within the larger field of functional analysis? In other words, does a better understanding of Banach spaces tell us anything about the operators between these spaces?
- ▹ Good mathematics doesn't exist in a vacuum. We also want to keep an eye out for applications of the theory of Banach spaces to "mainstream" analysis. Conversely, we will want to be on the lookout for applications of mainstream tools to the theory of Banach spaces. Among others, we will look for connections with probability, harmonic analysis, topology, operator theory, and plain ol' calculus. By way of an example, we might consider the calculus of "vector-valued" functions $f : [0, 1] \to X$, where X is a Banach space. It would make perfect

sense to ask whether f is of bounded variation, for instance, or whether $\int_0^1 \|f(x)\|\,dx < \infty$. We'll put these tantalizing questions aside until we're better prepared to deal with them.

Notes and Remarks

The space $C[a, b]$ is arguably the oldest of the examples presented here; it was Maurice Fréchet who offered the first systematic study of the space (as a metric space) beginning in 1906. The space ℓ_2 was introduced in 1907 by Erhard Schmidt (of the "Gram–Schmidt process"). The space that bears his name held little interest for Hilbert, by the way. Hilbert preferred the concrete setting of integral equations to the abstractions of Hilbert space theory.

Schmidt's paper is notable in that it is believed to contain the first appearance of the familiar "double-bar" notation for norms. Both the notation ℓ_2 and the attribution "Hilbert space," though, are due to Frigyes (Frederic) Riesz. In fact, Riesz introduced the L_p spaces, and he, Fréchet, and Ernst Fischer noticed their connections with the ℓ_p spaces. Although many of these ideas were "in the air" for several years, it was Banach who launched the first *comprehensive* study of normed spaces in his 1922 dissertation [5], culminating in his 1932 book [6]. For more on the prehistory of functional analysis and, in particular, the development of function spaces, see the detailed articles by Michael Bernkopf [13, 14], the writings of A. F. Monna [104, 105], and the excellent chapter notes in Dunford and Schwartz [42].

For much more on the classical and "neoclassical" Banach spaces, see the books by Adams [1], Bennett and Sharpley [12], Dunford and Schwartz [42], Duren [43], Lacey [88], Lindenstrauss and Tzafriri [93, 94, 95], and Wojtaszczyk [147]. For more on the history of open questions and unresolved issues in Banach space theory, see Banach's book [6], its review by Diestel [32], and its English translation with notes by Bessaga and Pelczyński [7]; see also Day [29], Diestel [33], Diestel and Uhl [34], Megginson [100], and the articles by Casazza [20, 21, 22, 23], Mascioni [99], and Rosenthal [124, 125, 126, 127].

Exercises

1. If $(X, \|\cdot\|)$ is any normed linear space, show that the operations $(x, y) \mapsto x + y$ and $(\alpha, x) \mapsto \alpha x$ are continuous (on $X \times X$ and $\mathbb{R} \times X$, respectively). [It doesn't much matter what norms we use on $X \times X$ and $\mathbb{R} \times X$; for example, $\|(x, y)\| = \|x\| + \|y\|$ works just fine. (Why?)] If Y is a (linear) subspace of X, conclude that its closure $\overline{Y}$ is again a subspace.

2. Show that $(X, \|\cdot\|)$ is complete if and only if every absolutely summable series is summable; that is, if and only if $\sum_{n=1}^\infty \|x_n\| < \infty$ always implies that $\sum_{n=1}^\infty x_n$ converges in (the norm of) X.
3. Show that $C^{(1)}[0,1]$, the space of functions $f : [0,1] \to \mathbb{R}$ having a continuous first derivative, is complete under the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$.
4. Show that s is complete under the Fréchet metric (1.3).
5. Show that ℓ_∞ is not separable.
6. Given $0 < p < 1$, show that $\|f + g\|_p \le 2^{1/p}(\|f\|_p + \|g\|_p)$ for $f, g \in L_p$. A better estimate (with a slightly harder proof) yields the constant $2^{(1/p)-1}$ in place of $2^{1/p}$.
7. Let $1 < p < \infty$ and let $1/p + 1/q = 1$. Show that for positive real numbers a and b we have $ab \le a^p/p + b^q/q$ with equality if and only if $a = b$. For $0 < p < 1$ (and $q < 0$!), show that the inequality reverses.
8. Let $0 < p < 1$ and let $1/p + 1/q = 1$. If f and g are nonnegative functions with $f \in L_p$ and $\int g^q > 0$, show that $\int fg \ge (\int f^p)^{1/p}(\int g^q)^{1/q}$.
9. Given $0 < p < 1$ and nonnegative functions $f, g \in L_p$, show that $\|f + g\|_p \ge \|f\|_p + \|g\|_p$.
10. Prove the string of inequalities (1.2) for $x \in \ell_1$.
11. Prove the string of inequalities (1.7) for $f \in L_\infty[0,1]$.
12. Given $1 \le p, q \le \infty$, $p \ne q$, show that $L_p(\mathbb{R}) \not\subset L_q(\mathbb{R})$.
13. Given a compact Hausdorff space X, show that $C_0(X)$ is a closed subspace of $C_b(X)$ and that $C_C(X)$ is dense in $C_0(X)$.
14. Let H be a separable Hilbert space with orthonormal basis (e_n) and let K be a compact subset of H. Given $\varepsilon > 0$, show there exists an N such that $\|\sum_{n=N}^\infty \langle x, e_n\rangle e_n\| < \varepsilon$ for every $x \in K$. That is, if K is compact, then these "tail series" can be made *uniformly* small over K.