THE DEVELOPMENT OF STRUCTURED RING SPECTRA

A. D. ELMENDORF

Abstract. The problem of giving a succinct description of multiplicative structure on spectra was recognized almost as soon as the idea of a spectrum was formulated. This paper aims to describe the major features of the historical precursors to the S-module approach of [2]. In particular, we consider the purely homotopical notion of a ring spectrum, May's concepts of external smash product and its internalization, the Lewis-May twisted half-smash product, and this product's use in formulating May and Quinn's notion of an E_∞ ring spectrum. We then describe how three essentially trivial (but crucial) observations led to the idea of an $\mathbb{L}$-spectrum, and soon thereafter to S-modules. We conclude by describing the good formal and homotopical properties of the category of S-modules.

The aim of this paper is to give some historical background to the first of the modern treatments of structured ring spectra: the S-module approach of [2]. There have been subsequent models developed as well; I'd like to mention in particular the symmetric spectra originally developed by Jeff Smith [3] and the orthogonal spectra of Mandell and May ([6] and [7]). There has been quite a lot of work done relating these various approaches, but this paper is concerned with the S-module approach only.

The invention of spectra, in the sense of algebraic topology, is usually credited to Lima in the late 1950's, although the first definition in print appears to be Spanier's [10]. There were a number of sources for the idea, of which I'd like to mention three: stable maps and the Spanier-Whitehead category, cohomology and Eilenberg-Mac Lane spaces, and cobordism. All of these involve sequences of spaces $A_0, A_1, \ldots$ and maps

$$A_i \to \Omega A_{i+1}$$

(or equivalently, $\Sigma A_i \to A_{i+1}$.) Two of the problems that were recognized early on were

(1) What is the "right" notion of morphism; some only "start to exist" after n stages, and
(2) what is the correct way to formulate multiplicative structure?

Boardman gave what were quickly recognized as the right answers **after** passage to homotopy – he constructed a symmetric monoidal closed triangulated category, now universally called **the** stable category, whose study has

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a large literature of its own. Although Boardman never published his construction, an account was given by Vogt [11]. Adams gave a treatment in his Chicago notes [1], and May also gave a construction in a series of papers and books; see particularly [8] and [9]. All these constructions shared a common problem: the smash product construction in the underlying category of spectra was not associative until passage to homotopy. As a consequence, there were no strict ring spectra, and no "good" categories of module spectra. One problem in particular will illustrate this: if R is a ring spectrum in the weak sense, i.e., it descends to a ring object in the stable category, and if M and N are R-modules in the same weak sense, and further we have a map $M \to N$ of R-modules, then the cofiber of this map need not even be an R-module. Although topologists were able to use these weak notions to good effect nonetheless, the situation was clearly less than completely satisfactory.

Progress came first from Peter May, who began by resolving problem 1; see [8]. His solution was to restrict attention to spectra for which the structure maps $A_i \to \Omega A_{i+1}$ are homeomorphisms. Having done so, he showed that all spectra are weakly equivalent to ones of this restrictive form, and further, the "naive" sort of morphism, consisting of sequences of maps making obvious diagrams commute, suffice for this sort of spectrum. (The modern point of view is that he restricts his attention to fibrant objects.) The next step was to remove the indexation on natural numbers, by developing what he called coordinate-free spectra ; see [9], although some details were later deleted in the equivariant version due to Lewis and May [4] (and ironically enough, restored in the definition of $\mathbb{L}$-spectra, below.) These are defined by first picking a **universe** $\mathcal{U}$: a real inner product space isomorphic to $\mathbb{R}^\infty$, topologized using the colimit topology from the sequence

$$\{0\} \subset \mathbb{R}^1 \subset \mathbb{R}^2 \subset \cdots ,$$

with this topology used to topologize spaces of linear isometries that will arise shortly. The index set for a spectrum over the universe $\mathcal{U}$ is the set of finite dimensional subspaces of $\mathcal{U}$. In detail, a spectrum E assigns a space EV to each finite dimensional subspace $V < \mathcal{U}$, and whenever $W \perp V$, there is a structure homeomorphism

$$EV \xrightarrow{\cong} \Omega^W E(V \oplus W),$$

subject to an associativity diagram. Here $\Omega^W X$ is the function space $F(S^W, X)$, and S^W is the one-point compactification of W. Morphisms from E to E' consist of maps $EV \to E'V$ making the obvious squares commute. We get the category $\mathcal{SU}$ of spectra over $\mathcal{U}$. We note for later use that $\mathcal{SU}$ is both complete and cocomplete, meaning it has all limits and colimits; this is not related to the coordinate-free nature of the spectra in $\mathcal{SU}$.

A key point about spectra with structure maps consisting of homeomorphisms is that the constituent spaces need only be given for a cofinal set of indices: the rest can be filled in by looping spaces given for larger indices. This allows us to define the **external smash product**, which is a functor $\mathcal{SU} \times \mathcal{SU}' \to \mathcal{S}(\mathcal{U} \oplus \mathcal{U}')$ for any pair of universes $\mathcal{U}$ and $\mathcal{U}'$. From the previous remark, we need only consider subspaces of $\mathcal{U} \oplus \mathcal{U}'$ of the form $V \oplus V'$ for $V < \mathcal{U}$ and $V' < \mathcal{U}'$, and for these, we make a preliminary definition of

$$(E \wedge E')(V \oplus V') := EV \wedge E'V'.$$

For structure maps, we use the composite

$$\begin{aligned}\Sigma^{W \oplus W'} EV \wedge E'V' \cong \Sigma^W EV \wedge \Sigma^{W'} E'V' &\to E(V \oplus W) \wedge E'(V' \oplus W') \\ &= (E \wedge E')(V \oplus V' \oplus W \oplus W').\end{aligned}$$

The main philosophical point here is that there is no issue with permuting indices, since all of the indexing subspaces are orthogonal, so the direct sum is completely independent of the order of the summands. There is one technical annoyance to be confronted: the adjoint structure maps

$$(E \wedge E')(V \oplus V') \to \Omega^{W \oplus W'}(E \wedge E')(V \oplus V' \oplus W \oplus W')$$

are not homeomorphisms any more. However, there is a "spectrification" functor that corrects this situation, and we do get a good smash product $E \wedge E'$ indexed on $\mathcal{U} \oplus \mathcal{U}'$, where by "good" I mean that we get a symmetric monoidal structure on the category

$$\coprod_{n \geq 0} \mathcal{S}(\mathcal{U}^n).$$

Unfortunately, the homotopy category we get is wrong: instead of the stable category, we get a coproduct of one copy of the unstable homotopy category (for $n = 0$) and infinitely many of the stable category. May's solution for this is to pick an element $f \in \mathcal{I}(\mathcal{U}^2, \mathcal{U})$ (the space of linear isometries from $\mathcal{U}^2$ to $\mathcal{U}$), and "push down" $E \wedge E'$ using f: we get a spectrum $f_*(E \wedge E') \in \mathcal{SU}$. (The push down process proceeds by defining $(f_*D)(V) = \Sigma^{V - ff^{-1}V} D(f^{-1}V)$, and then spectrifying.) Unfortunately, this destroys the associativity and commutativity of the external smash product, since no such choice of f is associative and commutative. May does show that this gives the right construction in homotopy.

The technical heart of the solution via S-modules is the twisted half-smash product introduced by Lewis, May, and Steinberger [4]. This is a functor with input two universes $\mathcal{U}$ and $\mathcal{U}'$, an unbased space A with a structure map $A \to \mathcal{I}(\mathcal{U}, \mathcal{U}')$, and a spectrum E over the first universe $\mathcal{U}$. The output is a spectrum $A \ltimes E$ over $\mathcal{U}'$. This construction has the following important formal properties:

(1) Given $A \to \mathcal{I}(\mathcal{U}, \mathcal{U}')$ and $B \to \mathcal{I}(\mathcal{U}', \mathcal{U}'')$, then there is a canonical isomorphism

$$B \ltimes (A \ltimes E) \cong (B \times A) \ltimes E.$$

(2) Given $A_1 \to \mathcal{I}(\mathcal{U}_1, \mathcal{U}_1')$ and $A_2 \to \mathcal{I}(\mathcal{U}_2, \mathcal{U}_2')$, then there is a canonical isomorphism

$$(A_1 \ltimes E_1) \wedge (A_2 \ltimes E_2) \cong (A_1 \times A_2) \ltimes (E_1 \wedge E_2).$$

(3) If $f \in \mathcal{I}(\mathcal{U}, \mathcal{U}')$, then $\{f\} \ltimes E \cong f_* E$. In particular, $\{\mathrm{id}_{\mathcal{U}}\} \ltimes E \cong E$.

In addition, the most important homotopical property of the twisted half-smash product is the following: given a homotopy equivalence $A_1 \to A_2$ (**not** a weak equivalence) and a structure map $A_2 \to \mathcal{I}(\mathcal{U}, \mathcal{U}')$, then the induced map

$$A_1 \ltimes E \to A_2 \ltimes E$$

is a homotopy equivalence when E is "tame"; this hypothesis is satisfied if E is a CW-spectrum. In general, we don't know if this map is even a weak equivalence. The point is that the homotopy equivalence between A_1 and A_2 need not be over $\mathcal{I}(\mathcal{U}, \mathcal{U}')$. For further details, see [2], especially the Appendix by Michael Cole. This homotopical property implies that the inclusion map $\{f\} \subset \mathcal{I}(\mathcal{U}^2, \mathcal{U})$ induces a homotopy equivalence

$$f_*(E \wedge E') \simeq \mathcal{I}(\mathcal{U}^2, \mathcal{U}) \ltimes (E \wedge E')$$

for tame spectra E and E'.

We are now in a position to describe the original notion of a structured ring spectrum, called an E_∞ ring spectrum. First, for notation, we let $\mathcal{L}(n) = \mathcal{I}(\mathcal{U}^n, \mathcal{U})$; this is the n^{th} space in the **linear isometries operad** using the universe $\mathcal{U}$. The intuition is that given a spectrum E, the spectrum $\mathcal{L}(n) \ltimes E^{\wedge n}$ encodes all possible n-fold smash powers of E, and has the correct homotopy type (at least when E is tame.)

Definition 1. *An E_∞ ring spectrum R over $\mathcal{U}$ is a spectrum over $\mathcal{U}$ together with structure maps*

$$\xi_n : \mathcal{L}(n) \ltimes R^{\wedge n} \to R.$$

These must be "coherent" in the sense that several diagrams must commute; the most important (and largest) is the following, in which the map γ is the structure map for the operad $\mathcal{L}$:

$$\begin{array}{ccc}
\mathcal{L}(n) \ltimes ((\mathcal{L}(j_1) \ltimes R^{\wedge j_1}) \wedge \cdots \wedge (\mathcal{L}(j_n) \ltimes R^{\wedge j_n})) & \xrightarrow{1 \ltimes (\xi_{j_1} \wedge \cdots \wedge \xi_{j_n})} & \\
\downarrow \cong & & \\
(\mathcal{L}(n) \times \mathcal{L}(j_1) \times \cdots \times \mathcal{L}(j_n)) \ltimes R^{\wedge (j_1 + \cdots + j_n)} & & \mathcal{L}(n) \ltimes R^{\wedge n} \\
\downarrow \gamma \ltimes id & & \downarrow \xi_n \\
\mathcal{L}(j_1 + \cdots + j_n) \ltimes R^{\wedge (j_1 + \cdots + j_n)} & \xrightarrow[\xi_{j_1 + \cdots + j_n}]{} & R.
\end{array}$$

Further, the maps must be commutative, in the sense that ξ_n descends to a map from the orbit spectrum $\mathcal{L}(n) \ltimes_{\Sigma_n} R^{\wedge n}$.

This definition can be given in an alternative, somewhat more formal way:

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Definition 2. *Given a spectrum E over $\mathcal{U}$, let*

$$\mathbb{C}E = \bigvee_{n\geq 0} \mathcal{L}(n) \ltimes_{\Sigma_n} E^{\wedge n}.$$

Then $\mathbb{C}$ is a monad in $\mathcal{SU}$, and an E_∞ ring spectrum is the same thing as a $\mathbb{C}$-algebra.

And there things stood for 15 or 20 years.

It wasn't until 1993 that a combination of three essentially trivial observations led to a breakthrough with the development of $\mathbb{L}$-spectra and, from them, S-modules; see [2] for full details. The first observation is that $\mathcal{L}(2) \ltimes (E \wedge E')$ is a **canonical** smash product for E and E', encoding all possible choices of f_*, and further it has the correct homotopy type. The stumbling block is that it's not associative. The key to correcting this defect comes from the second observation, which is that $\mathbb{C}$ has a tiny submonad $\mathbb{L}$, defined as

$$\mathbb{L}E := \mathcal{L}(1) \ltimes E.$$

An **$\mathbb{L}$-spectrum** is simply an algebra over the monad $\mathbb{L}$. Since $\mathbb{L}$ is a submonad of $\mathbb{C}$, it follows automatically that every E_∞ ring spectrum is an $\mathbb{L}$-spectrum. Further, although this took a bit of work, $\mathbb{L}$-spectra form a perfectly good model of the stable category.

The third observation, due to Mike Hopkins, tells us how to put the first two observations together in order to construct an associative smash product. It is:

Lemma 3. (Hopkins' Lemma) *Consider the left action of $\mathcal{L}(1)$ on $\mathcal{L}(j)$ for any j and the right action of $\mathcal{L}(1) \times \mathcal{L}(1)$ on $\mathcal{L}(2)$, both by means of composition. Then if $i \geq 1$ and $j \geq 1$, the structure map γ of the operad $\mathcal{L}$ induces an isomorphism*

$$\mathcal{L}(2) \times_{\mathcal{L}(1)\times\mathcal{L}(1)} \mathcal{L}(i) \times \mathcal{L}(j) \cong \mathcal{L}(i+j).$$

Proof. By choosing isomorphisms $\mathcal{U}^i \cong \mathcal{U}$ and $\mathcal{U}^j \cong \mathcal{U}$, the coequalizer splits. □

This allows us to make a key definition.

Definition 4. *Given $\mathbb{L}$-spectra M and N, their smash product is given by*

$$M \wedge_{\mathbb{L}} N := \mathcal{L}(2) \ltimes_{\mathcal{L}(1)\times\mathcal{L}(1)} (M \wedge N).$$

Here the unsubscripted smash product is the external smash product described above.

Proposition 5. *The smash product of $\mathbb{L}$-spectra is coherently associative and commutative.*

Proof. The essential point is the associativity, and this follows by using Hopkins' Lemma to show that both ways of associating are canonically isomorphic to

$$\mathcal{L}(3) \ltimes_{\mathcal{L}(1)^3} (M_1 \wedge M_2 \wedge M_3).$$

□

A small problem is that this smash product of $\mathbb{L}$-spectra is not quite unital; instead, there is a canonical weak equivalence

$$\lambda : S \wedge_{\mathbb{L}} M \to M$$

for any $\mathbb{L}$-spectrum M, but this is sufficient to formulate most concepts. In particular, we can define a strictly commutative $\mathbb{L}$-ring spectrum as an $\mathbb{L}$-spectrum A together with a unit map $\eta : S \to A$ and an associative, commutative, and unital map

$$\mu : A \wedge_{\mathbb{L}} A \to A.$$

It is now an easy proposition that this recovers exactly the definition of E_∞ ring spectrum!

However, we don't have to be satisfied with the weak notion of units present with $\mathbb{L}$-spectra, because of a stroke of good luck: it turns out that the unit map for the sphere spectrum, $\lambda : S \wedge_{\mathbb{L}} S \to S$, is an isomorphism. This is because of the "accident" that Hopkins' lemma is true when $i = j = 0$, although the proof is different (and the lemma fails when one index is 0 and the other is not.) It follows immediately that $\lambda : S \wedge_{\mathbb{L}} M \to M$ is an isomorphism precisely when M is of the form $S \wedge_{\mathbb{L}} M'$, and it is these M's that we call S**-modules**. A bit of extra work gives us the following:

Proposition 6. *The smash product of $\mathbb{L}$-spectra is symmetric monoidal on the full subcategory of S-modules, and this subcategory models the stable category with its smash product.*

We write the category of S-modules as $\mathcal{M}_S$.

We are now in a good position to mimic all the formal apparatus of commutative algebra, once a few more details are settled; it is to these we now turn. First, we would like to have a function spectrum construction adjoint to the smash product, just as one has in categories of (ordinary) modules. This relies ultimately on the fact that the twisted half-smash product has a right adjoint, called the twisted function spectrum, written $F[A, E')$, with input $A \to \mathcal{I}(\mathcal{U},\mathcal{U}')$ and a spectrum E' over $\mathcal{U}'$, and output a spectrum over $\mathcal{U}$. For details, see [2]. The end result is all we could expect: there is a function spectrum construction on S-modules, written $F_S(M, N)$ for which

$$\mathcal{M}_S(M \wedge_{\mathbb{L}} N, P) \cong \mathcal{M}_S(M, F_S(N, P)).$$

Our second piece of unfinished business is to show that $\mathcal{M}_S$ has all the limits and colimits we could possibly want.

Proposition 7. *The category of S-modules is complete and cocomplete.*

Proof. First, May's category $\mathcal{SU}$ is complete and cocomplete, and the category of $\mathbb{L}$-spectra is a category of algebras over it. Therefore, by [5], section VI.2, exercise 2, the category of $\mathbb{L}$-spectra is complete. Further, $\mathbb{L}$ has a right adjoint $\mathbb{L}^\#$, given by $\mathbb{L}^\# E := F[\mathcal{L}(1), E)$. $\mathbb{L}^\#$ is consequently a comonad and the category of algebras over $\mathbb{L}$ can be identified with the category of coalgebras over $\mathbb{L}^\#$. By the dual exercise, $\mathbb{L}$-spectra form a cocomplete category. Next, we examine the functor $S \wedge_{\mathbb{L}} \_\_ : \mathbb{L}\text{-spectra} \to \mathcal{M}_S$, and find that it has both a left and a right adjoint, with the right adjoint being the inclusion of $\mathcal{M}_S$ into $\mathbb{L}$-spectra. Consequently, colimits in $\mathcal{M}_S$ are created in $\mathbb{L}$-spectra, and limits exist and are gotten by applying $S \wedge_{\mathbb{L}} \_\_$ to the limit in $\mathbb{L}$-spectra. □

Now we can import the entire formal apparatus of commutative algebra into stable homotopy theory: rings, commutative rings, algebras, left and right modules, tensor products, and function objects, with all the expected properties. As an example, we define a commutative S-algebra to be simply a commutative monoid in the symmetric monoidal category of S-modules, and we quickly see that all of them are E_∞ ring spectra, and further, given an E_∞ ring spectrum A, then $S \wedge_{\mathbb{L}} A$ is a commutative S-algebra. Because of the isomorphism $S \wedge_{\mathbb{L}} S \cong S$, this accounts for all commutative S-algebras, and since the unit map $\lambda : S \wedge_{\mathbb{L}} A \to A$ is always a weak equivalence, and easily seen to be a map of E_∞ ring spectra, we recover all the homotopical properties of E_∞ ring spectra by considering only commutative S-algebras.

I'd like to close by mentioning additional structure that is present: the categories of S-modules, S-algebras, commutative S-algebras, and the categories of algebras and modules over a given S-algebra are all topological model categories in which all objects are fibrant. In some sense this is the most exciting part of the new developments with structured ring spectra, since it allows us to talk about homotopy categories that were not even in the picture previously. These new homotopy categories have already inspired a considerable body of work, some of which appears elsewhere in this volume, but clearly much more is still to be done. Let's look forward to the exploration of these brave new worlds!

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COMPROMISES FORCED BY LEWIS'S THEOREM

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Abstract. In 1991, Gaunce Lewis published a theorem showing that a quite minimal list of desiderata for an "ideal" category of spectra was inconsistent; see [4]. This result requires any category modeling stable homotopy theory to make some compromises in its formal structure. This short paper describes the compromises present in $\mathcal{M}_S$, the category of S-modules developed in [2], together with the amusing consequence that $\mathcal{M}_S$ contains a copy of the (unstable!) category of topological spaces.

At this point we have several categories of spectra that are symmetric monoidal, with their smash products descending to the smash product in the stable category; let me mention in particular the S-modules of [2] and the symmetric spectra of [3]. These categories are much more nicely behaved than any of their predecessors, but their behavior is not absolutely ideal, because it can't be. This is a theorem of Gaunce Lewis's, whose paper [4] was published before any of the current batch of symmetric monoidal categories of spectra were developed. Suppose we have a candidate for a "good" category of spectra, which we ambiguously call $\mathcal{S}$. Lewis sets out the following pretty minimal list of properties for $\mathcal{S}$, all of which are devoutly to be desired:

(1) The category $\mathcal{S}$ has a symmetric monoidal product, which we call smash and write $\wedge$, as usual.
(2) Let $\mathcal{T}$ be the category of based topological spaces (in some convenient version such as compactly generated weak Hausdorff). Then there is a pair of functors $\Sigma^\infty : \mathcal{T} \to \mathcal{S}$ and $\Omega^\infty : \mathcal{S} \to \mathcal{T}$ with Σ^∞ being left adjoint to Ω^∞.
(3) The unit for the smash product in $\mathcal{S}$ is $\Sigma^\infty S^0$.
(4) Σ^∞ is a lax monoidal functor in the sense that there is a natural map

$$\Sigma^\infty(X \wedge Y) \to \Sigma^\infty X \wedge \Sigma^\infty Y,$$

subject to diagrams encoding commutation with the monoidal structure maps.

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(5) There is a natural weak equivalence $\theta : \Omega^\infty\Sigma^\infty X \to QX$ (where QX is the usual stabilization construction) for which the diagram

$$\begin{array}{ccc} & X & \\ {\scriptstyle\eta}\swarrow & & \searrow{\scriptstyle\eta} \\ \Omega^\infty\Sigma^\infty X & \xrightarrow{\ \theta\ } & QX \end{array}$$

commutes, where η is used generically for the unit of an adjunction.

Theorem 1. (Lewis '89) *The above five properties are inconsistent.*

The proof is distressingly simple. Equivalent to property 4, there is a natural map

$$\Omega^\infty E_1 \wedge \Omega^\infty E_2 \to \Omega^\infty(E_1 \wedge E_2)$$

which also commutes with the monoidal structure maps. Suppose E is a commutative monoid in $\mathcal{S}$: what we would like to call a strictly commutative ring spectrum. Then the two maps

$$\Omega^\infty E \wedge \Omega^\infty E \longrightarrow \Omega^\infty(E \wedge E) \xrightarrow{\Omega^\infty\mu} \Omega^\infty E$$

and

$$S^0 \xrightarrow{\eta} \Omega^\infty\Sigma^\infty S^0 \longrightarrow \Omega^\infty E$$

make $\Omega^\infty E$ into a commutative monoid in $\mathcal{T}$, using the symmetric monoidal smash product of based spaces. Now the unit, in this case $\Sigma^\infty S^0$, is always a commutative monoid in a symmetric monoidal category, so in particular, $\Omega^\infty\Sigma^\infty S^0$ must be a commutative monoid in $\mathcal{T}$. From property 5, we now see that QS^0 is weakly equivalent to a commutative monoid. It follows from a theorem of Moore [5] that QS^0 is homotopic to a product of Eilenberg-Mac Lane spaces. Life would be a lot simpler if this were true...

As a consequence of this theorem, every "good" category of spectra has to edge around the fact that it can't satisfy all five of these properties simultaneously. Here's what $\mathcal{M}_S$, the category of S-modules does.

$\mathcal{M}_S$ is symmetric monoidal, so property 1 is satisfied, and there is an adjoint pair $(\Sigma^\infty, \Omega^\infty)$, so 2 is satisfied. We actually have an isomorphism

$$\Sigma^\infty(X \wedge Y) \cong \Sigma^\infty X \wedge_S \Sigma^\infty Y,$$

so property 3 is more than satisfied. And the unit for $\wedge_S$ is $\Sigma^\infty S^0$, so property 4 is satisfied, too. This leaves property 5 to fail, which it does in spectacular fashion: $\Omega^\infty\Sigma^\infty X$ is actually homeomorphic to X for all spaces X, no matter how badly behaved! Clearly Ω^∞ is not what we usually think, since Σ^∞ does look pretty much like what we think it should. In fact, what I've been calling $\Omega^\infty E$ is actually the space $\mathcal{M}_S(S, E)$ for an S-module E. The fact that $\mathcal{M}_S(S, \Sigma^\infty X) \cong X$ is a special case of the following theorem; see [1] for details:

Theorem 2. *The functor $\Sigma^\infty : \mathcal{T} \to \mathcal{M}_S$ induces a homeomorphism*

$$\mathcal{T}(X, Y) \to \mathcal{M}_S(\Sigma^\infty X, \Sigma^\infty Y)$$

for all spaces X and Y.

The proof reduces the general case to the specific one first mentioned, and then computes in an extremely explicit fashion.

As a consequence of this theorem, $\mathcal{M}_S$ has inside it a perfect copy of $\mathcal{T}$, the category of topological spaces, as the full subcategory of suspension spectra. This should seem bizarre, since the purpose of $\mathcal{M}_S$ is to model stable homotopy, and in $\mathcal{T}$, nothing has been stabilized. In fact, if we just use honest homotopy classes, which amounts to taking π_0 of the mapping spaces between spectra, we don't get stable homotopy, as we see from the presence of this copy of $\mathcal{T}$. The situation is saved by the requirement that we invert the weak equivalences, not just the ordinary homotopy equivalences, and doing so precisely stabilizes the maps between suspension spectra. See [1] for the details. As an added amusement, we find that if X is a CW complex, then in the model category structure on $\mathcal{M}_S$, the S-module $\Sigma^\infty X$ is homotopic to a cofibrant S-module precisely when X is contractible. From this point of view, it's cofibrant replacement that stabilizes the maps between suspension spectra.

In conclusion, I should mention that there is another candidate for the $(\Sigma^\infty, \Omega^\infty)$ adjunction between $\mathcal{T}$ and $\mathcal{M}_S$ which does satisfy $\Omega^\infty\Sigma^\infty X \simeq QX$. Obviously, this is the correct pair of functors to use when doing homotopy theory. However, if we use this pair, we find that $\Sigma^\infty S^0$ is not the unit of the smash product of S-modules, this being crucial to the proof of Lewis's theorem. Once again, we have to compromise when setting up the formal properties of a "good" category of spectra.

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