Combinatorics of finite geometries

This book is an introductory text on the combinatorial theory of finite geometry. It assumes only a basic knowledge of set theory and analysis, but soon leads the student to results at the frontiers of research. It begins with an elementary combinatorial approach to finite geometries based on finite sets of points and lines, and moves into the classical work on affine and projective planes. This is followed by chapters dealing with polar spaces, partial geometries, and generalized quadrangles. The second edition contains an entirely new chapter on blocking sets in linear spaces, which highlights some of the most important applications of blocking sets – from the initial game-theoretic setting to their very recent use in cryptography. Extensive exercises at the end of each chapter ensure the usefulness of this book for senior undergraduate and beginning graduate students.

LYNN MARGARET BATTEN

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Second edition

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TO MY SISTER AUDREY

'I have opened all the doors in my head.'

Marilyn French, *The Women's Room*

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Preface

The principal changes to the second edition occur in sections 4.5 (which has been completely re-done) and 6.2 (which has been brought up to date). I want to thank particularly Stan Payne for his help in reviewing the appropriate literature for inclusion in 6.2.

In addition, a new chapter, chapter 8, on blocking sets, has been added. Blocking sets have numerous applications in game theory and in testing of statistical designs. We also describe some very recent applications to cryptography.

I wish to thank the University of Manitoba at this time for its support in the production of this new edition.

LMB

Preface to the first edition

This text is designed as an introduction to finite geometry for the undergraduate student. It could be used for second or third year students with some aptitude for, but not necessarily a great deal of background in, mathematics. A second year general student would have a good foundation in synthetic geometry with the completion of the first four chapters. A third year honours student or a fourth year student could be expected to complete the book in a one year course.

As far as background is concerned, only a fundamental knowledge of functions and set theory to the first year university level is essential. Some linear algebra and field theory would be useful for some parts of chapters 5, 6 and 7, but at a minimal level.

Listed in the last section of each chapter are forty to fifty exercises. Some of these are designed to consolidate the student's acquaintance with the concepts presented in the chapter. Others give additional results and introduce new concepts. The exercises form an important part of the material and I would strongly advise that students be assigned several each week. The very difficult problems have been starred.

The material presented in the book is based on the notion of 'connection number' in a near-linear space. Given a system of points and lines, for any point p not on a line ℓ, the connection number, $c(p,\ell)$, is the number of points on ℓ which are connected to p by a line. In chapter 1 (near-linear spaces) there are no restrictions on $c(p,\ell)$. In chapter 2 (linear spaces), $c(p,\ell)$ must always be the total number of points on ℓ. Chapters 3 and 4 look at special cases of the systems discussed in chapter 2. These are the classical affine and projective spaces. Chapters 5, 6 and 7 deal with structures which have been introduced more recently; in chapter 5, $c(p,\ell)$ must always be 1 or the total number of points on ℓ. Chapter 6 deals more particularly with those structures for which $c(p,\ell)$ is always 1. Finally, in chapter 7 we look at near-linear spaces for which $c(p, \ell)$ is always a fixed positive integer.

In North America, the material covered in chapters 2, 3 and 4 is generally acknowledged to be the 'classical' synthetic geometry. However, the geometries of chapters 5, 6 and 7 are already a part of the curricula of Western European schools and, in view of the important results of researchers on these geometries, it seems likely that in the near future polar spaces and partial geometries will be an intrinsic part of the North American programme.

I am indebted to many friends and colleagues for their kindness and support while this book was in progress. A special thanks to those who took time to proof-read and to make important suggestions: Francis Buekenhout, Frank de Clerck, James Hirschfeld, Stan Payne, Clare and Tom Ralston. To one of my students, Elizabeth Teitsma, a warm thank-you for providing me with diagrams of projective and affine planes, and also for proof-reading. A final thank-you to the second year Synthetic Geometry class at the University of Winnipeg, 1980–1, who put up with the disorganization that is part of putting a text together, and who supplied me with comments as to improvements of proofs, and levels of difficulty of problems.

I wish to acknowledge the University of Winnipeg for its financial support for typing of the final draft of the manuscript. And to Betty Harder who typed this final draft, my deep appreciation for the excellent and very professional finished product.

LMB