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Lynn Margaret Batten
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1

Near-linear spaces

I suppose you know that students of geometry and arithmetic and so forth begin by taking for granted odd and even, and the usual figures, and the three kinds of angles, and things akin to these, in every branch of study; they take them as granted and make them assumptions or postulates, and they think it unnecessary to give any further account of them to themselves or others, as being clear to everybody. Then, starting from these, go on through the rest by logical steps until they end at the object which they set out to consider.

Plato *The Republic* Book VI

1.1 Some basic concepts: consistency and dependence

The understanding throughout this book is that we work with a set P whose elements are called *points*, and a set L of certain subsets of P, whose elements are called lines.† We remind the reader that by definition a set has *distinct* elements.

A *space* $S=(P,L)$ is a system of points P and lines L such that certain conditions or *axioms* are satisfied. We can then consider two points of view: given a system of axioms about points and lines, can we find any spaces which satisfy it, or, given a familiar space (for example, real 3-space), what system or systems of axioms can be used to define it? We are only interested here in the former of these two questions.

As we shall be working a great deal with axiom systems, we discuss some of their properties in this section.

An axiom system is said to be *consistent* if it is possible to construct an example of a structure satisfying all the axioms. Otherwise the system is said to be *inconsistent*.

Consider the following examples where we suppose always that we are working with a system of points and lines.

Example 1.1.1

1. There are five points and six lines.
2. Each point is in one line.
3. Each line contains one point.

† The words *variety* and *block* are often used instead of *point* and *line* and later we shall introduce v and b for the number of points and lines respectively.

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Before checking this system for consistency or inconsistency, we note the following convention. When we say there are five points, we always mean that there are *precisely* five points. Otherwise we shall add 'at least' or 'at most' or some equivalent expression.

After spending some time trying to construct an example satisfying these three axioms, it becomes apparent that we are trying to set up a one-to-one correspondence (by 2 and 3) between sets with five points and six points respectively. This of course is impossible, so the system is in fact an inconsistent one.

Example 1.1.2

1. There are seven points and seven lines.
2. Each line has three points.
3. Each point is on three lines.

The fact that one has just spent an hour trying to find an example of such a space, with no luck, does not necessarily mean that no example exists. In fact, figure 1.1.1 represents an example of just such a system. If the points are labelled 0, 1, 2, 3, 4, 5, and 6, we can choose as lines the sets {1, 2, 4}, {2, 3, 5}, {3, 4, 6}, {0, 4, 5}, {1, 5, 6}, {0, 2, 6} and {0, 1, 3}.

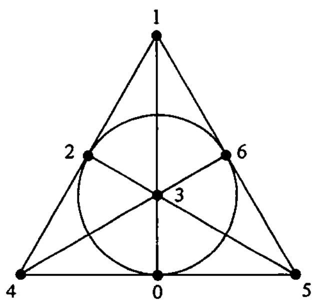

Figure 1.1.1.

Of course the labelling is arbitrary, but our choice here has an interesting property. Each line is of the form $\{1+i, 2+i, 4+i\}$ where i ranges between 0 and 6, and we always use the remainder on division by 7. That is, the set $\{1+5, 2+5, 4+5\} = \{6, 7, 9\}$ becomes the line $\{6, 0, 2\}$. This (near-linear) space is called the *Fano plane*.

So, to prove that a system is inconsistent, we must supply a formal proof of the fact. Whereas to prove consistency it suffices to provide an example.

A system of axioms is said to be *dependent* if one or more axioms can

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be proved using the remaining axioms. Otherwise the system is *independent*. We note that to prove dependency, it is enough to show that a particular axiom follows from the others. To prove independency, one must show that no axiom follows from the others.

Example 1.1.3

1. There are six points and four lines.
2. Each line has two points.
3. Each point is on at most four lines.

We claim that this system is dependent. In fact, it is easy to see that axiom 3 follows immediately from axiom 1: if there are only four lines, then clearly no point can be on more than four lines.

An alternative approach to proving dependence is to list all examples of systems satisfying axioms 1 and 2. If they all satisfy axiom 3, then axiom 3 follows from axioms 1 and 2.

To list all such examples, we use a systematic approach. First list all examples in which some point is on the maximum possible number of lines: four. Then list all examples for which some point is on three lines, but no point is on four lines; and so on. Diagrams representing these systems can be found in figure 1.1.2.

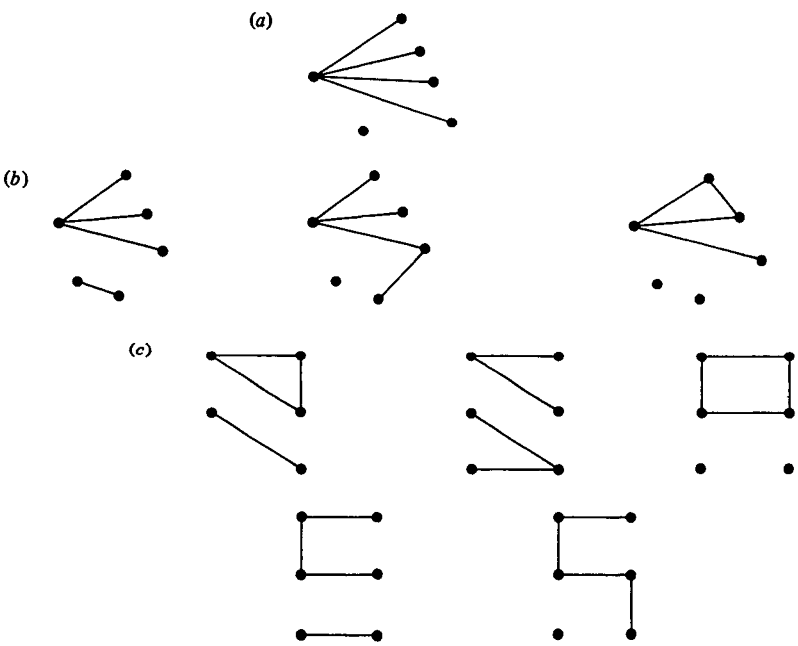

Figure 1.1.2.

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Example 1.1.4

1. There are four points and four lines.
2. Every point is on two lines.

This axiom system is quickly seen to be independent, as we can provide an example of a space satisfying 1 but not 2, and an example of a space satisfying 2 but not 1.

Let $P=\{1, 2, 3, 4\}$ and $L=\{\{1, 2\}, \{2, 3\}, \{1, 3\}, \{2, 4\}\}$. This satisfies 1 but not 2.

Let $P=\{1, 2, 3\}$ and $L=\{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$. This satisfies 2 but not 1.

Example 1.1.3 brings to the fore the following problem: when are two spaces the same? Intuitively we shall say that they are the same if a diagram for one can be twisted into a diagram for the other without 'breaking' any lines and without adding any extra 'joins'. Consider figure 1.1.3. The examples (*a*) and (*b*) are the same as we can move the line $\{2, 3\}$ around until the point 3 is on the other side of the line $\{1,4\}$. However, neither is the same as (*c*), because the points 3 and 4 would have to be joined together as one point.

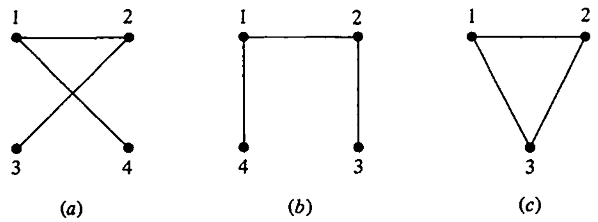

Figure 1.1.3.

We shall be coming back to this problem in section 1.7 where we shall define formally what we mean by two spaces being 'the same'.

1.2 Near-linear spaces

We restrict ourselves henceforth to spaces with certain properties. In particular, we shall attribute to lines some of the characteristics usually associated with them. Hence the following definition.

A *near-linear space*† is a space $S=(P, L)$ of points P and lines L such that

NL1 any line has at least two points, and

NL2 two points are on at most one line.

If p and q are distinct points which are on a line, then this line is unique

† Also called *partial plane*, a term introduced by M. Hall (1943).

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by NL2. We denote this unique line by pq. It should be clear then, that if r and s are any distinct points on the line pq, it must be the case (by NL2) that $pq = rs$.

Example 1.2.1. Let P be the set of points of Euclidean (real) 3-space, and let L be the set of all usual lines. Then (P, L) is a near-linear space.

Example 1.2.2. Let P be as in example 1.2.1 but let L be the set of all usual planes in 3-space. This is not a near-linear space as two points are on many planes.

Example 1.2.3. Let $P = \{1, 2, 3, 4, 5, 6\}$ and $L = \{\{1, 2, 3\}, \{2, 4\}, \{3, 4, 5\}, \{1, 4\}\}$. (See figure 1.2.1.) Then (P, L) is a near-linear space.

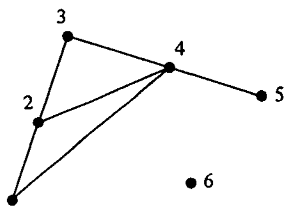

Figure 1.2.1.

Example 1.2.4. Let $P = \{1, 2, 3\}$, $L = \{\emptyset, \{1, 2\}, \{1, 2, 3\}\}$. (See figure 1.2.2 – it is rather difficult to draw the empty set.)

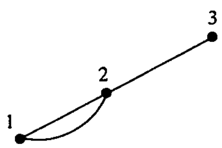

Figure 1.2.2.

This is clearly not a near-linear space. It violates both axioms in fact.

The reader can check that the spaces of figures 1.1.1, 1.1.2 and 1.1.3 are all near-linear spaces.

The definition of near-linear space does not assume that there are any points at all. In this case, there are no lines, and the near-linear space is denoted by $\emptyset$, the empty set. Furthermore, even if there are points, the definition does not imply that there are lines.

A word here about notation. In general, a point will be labelled $1, 2, \ldots$ or $a, b, \ldots$ in examples, but in proofs, $p, q, \ldots$ will be more common. To label lines, we use ℓ, and sometimes m and h. Often subscripts will be

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used. These are merely guidelines however, and we do not promise to stick to them entirely.

For the *order*, or number of elements of a set X, we use $|X|$.

Near-linear spaces have properties other than NL1 and NL2 in common, as we shall see by the following lemmas.

Lemma 1.2.1. *Two distinct lines of a near-linear space intersect in at most one point.*

Proof. Suppose ℓ_1 and ℓ_2 are distinct lines. If $|\ell_1 \cap \ell_2| \geq 2$, we contradict NL2. □

Lemma 1.2.2. *If ℓ_1 and ℓ_2 are such that $\ell_1 \subseteq \ell_2$, then $\ell_1 = \ell_2$.*

Proof. By NL1, ℓ_1 has at least two points, so that, by NL2, $\ell_1 = \ell_2$. □

Before moving into the next section, we introduce more notation. For the number of points in a near-linear space we use v, and for the number of lines, b.

For a line ℓ, $v(\ell)$ will be the number of points on ℓ or, equivalently, $|\ell|$.

For a point p, $b(p)$ will be the number of lines on p.

Note that these numbers may be infinite.

When talking about points and lines, we shall use the ordinary language of geometry: for example, points are *on* (rather than *in*) or *incident with* lines, lines are *on* or *incident with* points, points are *joined* by lines, etc.

1.3 New near-linear spaces from old

In this section we consider the construction of a new near-linear space from a given one.

Let $S = (P, L)$ be a near-linear space. We define a new near-linear space $R = (P', L')$ as follows. P' is an arbitrary subset of P and L' is the set of intersections $\ell \cap P'$ for any ℓ in L with at least two points in P'. It is an easy exercise to check that R is indeed a near-linear space. R is called a *restriction* of S and, in particular, the *restriction of S to P'*.

Example 1.3.1. Let P be the set of points of Euclidean 2-space, and L the usual lines. Define P' to be those points of P inside the unit circle centred at the origin. That is, a point (x, y) is in P' if $x^2 + y^2 < 1$. Any line of L meets P' in many points or misses it entirely. (The reader is asked to verify this fact.) Hence lines of L' are obtained from lines of L which meet P', restricted to the set of points in P'.

Example 1.3.2. Let $P = \{1,2,3,4,5\}$, $L = \{\{1,4\}, \{2,3\}, \{1,2,5\}\}$. If $P' = \{2,3,4\}$, then $L' = \{\{2,3\}\}$.

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As the choice of P' is made arbitrarily, we get a restriction corresponding to every subset of P. Hence, if v is finite, the number of restrictions might be 2^v. (Recall that a set of v elements has 2^v subsets.) We ask, then, if all restrictions of a given space are necessarily different where, by 'different', we mean our intuitive sense of the word, described at the end of section 1.1. We answer the question in the negative by considering the following example.

Example 1.3.3. $P=\{1,2,3,4,5\}$ and each pair of points is a line. (See figure 1.3.1.) Here the 'different' restrictions are $\varnothing$, a point, a line, a triangle, a square with its diagonals, and the whole space. So there are only five new near-linear spaces obtained out of a possible $2^5=32$.

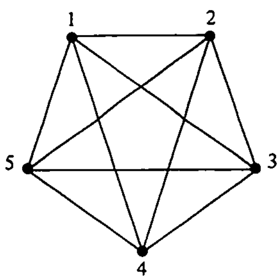

Figure 1.3.1.

Let $S=(P,L)$ be a near-linear space. We define the *dual (near-linear) space* $R=(P',L')$ of S as follows

$$P'=L$$

and any set of at least two lines of S which is the set of all lines through a fixed point of S is a line of L', and these are the only lines. In brief, $L'=\{\{p_1,\ldots,p_m\}|\ p_i\in P',\ m\geq 2$ and $p_1,\ldots,p_m$ are all the lines of S incident with a fixed point$\}$.

We illustrate this definition with the near-linear space of figure 1.2.1 where the lines may be labelled as follows: $\ell_1=\{1,2,3\}$, $\ell_2=\{3,4,5\}$, $\ell_3=\{1,4\}$, $\ell_4=\{2,4\}$. Then $P'=\{\ell_1,\ell_2,\ell_3,\ell_4\}$. Now for each point of S on at least two lines we obtain a line of L': $L'=\{\{\ell_1,\ell_3\},\{\ell_1,\ell_4\},\{\ell_1,\ell_2\},\{\ell_2,\ell_3,\ell_4\}\}$. The dual space is illustrated in figure 1.3.2.

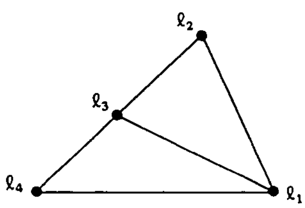

Figure 1.3.2.

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Lemma 1.3.1. *The dual space of a near-linear space is a near-linear space.*

Proof. By definition, any line in the dual space has at least two points so that NL1 is satisfied.

Consider two points of the dual space and let ℓ_1 and ℓ_2 be the lines of the near-linear space $S=(P,L)$ to which these two points correspond. Each line joining ℓ_1 and ℓ_2 in the dual space corresponds to a point of intersection of ℓ_1 and ℓ_2 in S and, since there is at most one such point of intersection by lemma 1.2.1, there is at most one line on ℓ_1 and ℓ_2 in the dual space. □

A *graph* is a near-linear space in which every line has precisely two points. The near-linear spaces of figures 1.1.3 and 1.3.1 are graphs.

It is possible to obtain a graph from a near-linear space in almost the same way as we obtain the dual space. Let $S=(P,L)$ be a near-linear space; the *line graph* $R=(P',L')$ is defined by

$$P' = L,$$
$$L' = \{\{p,q\} \mid \text{ where } p \text{ and } q \text{ are distinct intersecting lines of } L\}.$$

It is clearly a graph.

The line graph of figure 1.2.1, labelling the lines as above, is shown in figure 1.3.3.

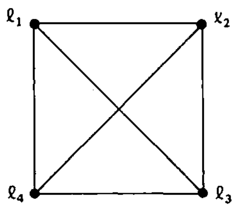

Figure 1.3.3.

The line graph of figure 1.1.2 (*a*) is also the graph of figure 1.3.3.

A *subspace* of a near-linear space (P,L) is a set X of points of P such that whenever p and q are points of X which are on a line pq of L, then the entire line pq is in X. The empty set, any point, any line and the whole space itself are always subspaces of a given space. The subspaces of figure 1.1.3 (*a*) are the ones mentioned above along with the sets $\{1,2,3\}$, $\{1,2,4\}$ $\{1,3\}$, $\{2,4\}$, $\{3,4\}$, $\{1,3,4\}$ and $\{2,3,4\}$. The subspaces of figure 1.1.1 are *only* the ones mentioned above.

A subspace becomes a near-linear space (check!) if we consider it to be the set of its points *and* of the lines of the space which belong to the subspace.

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Lemma 1.3.2. *The intersection of any number of subspaces is a subspace.*

Proof. Let X be the intersection of any number of subspaces. We need only show that, if p and q are points of X and p and q are on a line pq, then $pq \subseteq X$. But any subspace containing X then contains p and q, and so by definition pq. Therefore the line pq is in all subspaces of which X is the intersection, and so pq is a subset of X. □

1.4 Dimension

Let X be any set of points of a near-linear space $S=(P,L)$. The *closure* of X is a subspace which contains X but does not properly contain any subspace on X.

It is not obvious from the definition that the closure of X is unique, but this follows from lemma 1.4.1 below. The closure of X is thus the *smallest* subspace containing X.

Notation. $\langle X \rangle$ will denote the closure of the set X.
We first of all consider some examples.
In figure 1.1.1, $\langle\{5,6\}\rangle=\{1,5,6\}$ and $\langle\{0,3,4\}\rangle=S$.
In figure 1.2.1, $\langle\{1,2,5\}\rangle=\{1,2,3,4,5\}$.

In example 1.2.1, the closure of any set of at least two collinear points is the line on them. The closure of any set of four points, no three collinear, is the whole space.

In any near-linear space S, $\langle\varnothing\rangle=\varnothing$, $\langle p\rangle=p$ and $\langle S\rangle=S$. Also, for any set of points X, $X\subseteq\langle X\rangle, \langle X\rangle=\langle\langle X\rangle\rangle$ and if $X\subseteq Y$ then $\langle X\rangle\subseteq\langle X\rangle$.

Lemma 1.4.1. *The closure of a set X is the intersection of all subspaces on X.*

Proof. By lemma 1.3.2 this intersection is a subspace. It is easy to see that it is the smallest subspace on X as any subspace on X is included when we take the intersection. □

We say that X *generates* its closure. Conversely, given a subspace R we say that X is a *generating set* for R if $\langle X\rangle=R$, so that, also, X generates R.

In section 1.1 we defined an independent axiom system as one in which no axiom followed from the others. So no superfluous information is given. In the same vein, we wish to make a definition of independence for subsets of a near-linear space. An independent set will be one which has 'just enough' points to generate its closure.

An *independent set* X is a set of points such that for each $x\in X$, $x\notin\langle X\backslash\{x\}\rangle$.

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Again, let us illustrate this definition using figure 1.1.1. In the near-linear space there, the set $X = \{1, 5, 6\}$ is *not* independent. It contains more than enough points to generate its closure, which is itself. The points 1 and 5 would suffice:

$$6 \in \langle X \backslash \{6\} \rangle.$$

We call such a set *dependent*.
The set $X = \{1, 5\}$ *is* independent:

$$1 \notin \langle X \backslash \{1\} \rangle = \{5\} \quad \text{and} \quad 5 \notin \langle X \backslash \{5\} \rangle = \{1\}.$$

In figure 1.2.1, the sets $\{5, 6\}$, $\{4, 5, 6\}$ and $\{2, 4, 5, 6\}$ are independent. The set $\{1, 2, 4, 5\}$ is dependent.

In figure 1.3.1 the set $\{1, 2, 3, 4, 5\}$ is independent.

It is not difficult to see that the empty set is independent (there are no points to check!), and that a single point is always independent. So is any pair of points.

A *basis* of a near-linear space S is an independent subset of the points of S which generates S.

A basis is not necessarily unique. The space of figure 1.1.1 has $\{1, 2, 0\}$ and $\{3, 6, 5\}$ as bases (and many more). Any basis of the space of figure 1.2.1 must contain the point 6. $\{1, 2, 4, 6\}$ is a basis for example. There is precisely one basis for figure 1.3.1, namely $\{1, 2, 3, 4, 5\}$.

For a given near-linear space, do all bases have the same number of elements? The answer is no, as can be seen by considering the example of figure 1.4.1. The set $\{1, 2, 3\}$ is a basis while so is the set $\{4, 5, 6, 7\}$.

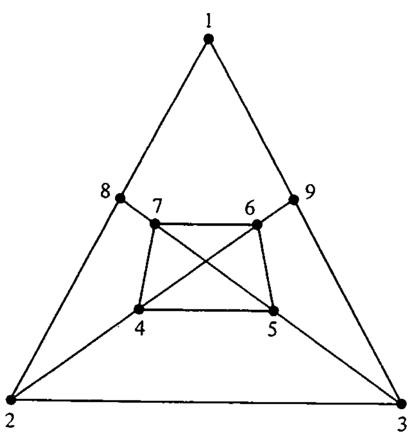

Figure 1.4.1.

We wish now to define the dimension of a finite near-linear space (i.e., one with a finite number of points) in terms of the number of elements

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