I

Artin rings

While we are assuming that the reader is familiar with general concepts of ring theory, such as the radical of a ring, and of module theory, such as projective, injective and simple modules, we are not assuming that the reader, except for semisimple modules and semisimple rings, is necessarily familiar with the special features of the structure of artin algebras and their finitely generated modules. This chapter is devoted to presenting background material valid for left artin rings, and the next chapter deals with special features of artin algebras. All rings considered in this book will be assumed to have an identity and all modules are unitary, and unless otherwise stated all modules are left modules.

We start with a discussion of finite length modules over arbitrary rings. After proving the Jordan–Hölder theorem, we introduce the notions of right minimal morphisms and left minimal morphisms and show their relationship to arbitrary morphisms between finite length modules. When applied to finitely generated modules over left artin rings, these results give the existence of projective covers which in turn gives the structure theorem for projective modules as well as the theory of idempotents in left artin rings. We also include some results from homological algebra which we will need in this book.

1 Finite length modules

In this section we introduce the composition series and composition factors for modules of finite length. We prove the Jordan–Hölder theorem and give an interpretation of it in terms of Grothendieck groups.

Let Λ be an arbitrary ring. Given a family of Λ-modules $\{A_i\}_{i\in I}$ we denote by $\coprod_{i\in I} A_i$ the **sum** of the A_i in the category of Λ-modules. The reader should note that direct sum is another commonly used terminology for what we call sum, and another notation is $\oplus_{i\in I} A_i$. We recall that a Λ-module A is **semisimple** if A is a sum of simple Λ-modules and that Λ is a **semisimple ring** if Λ is a semisimple Λ-module.

A basic characterization of such modules is that A is semisimple if and only if every submodule of A is a summand of A. As a consequence, every submodule and every factor module of a semisimple module are again semisimple. But in general the category of semisimple modules, or finitely generated semisimple modules, is not closed under extensions. This leads to the study of modules of finite length, which is the smallest category closed under extensions which contains the simple modules.

A module A is said to be of **finite length** if there is a finite filtration of submodules $A = A_0 \supset A_1 \supset \cdots \supset A_n = 0$ such that A_i/A_{i+1} is either zero or simple for $i = 0, \ldots, n-1$. We call such a filtration F of A a **generalized composition series**, and the nonzero factor modules A_i/A_{i+1} the **composition factors** of the filtration F. If no factor module A_i/A_{i+1} is zero for $i = 0, \ldots, n-1$, then F is a **composition series** for A. For a simple Λ-module S we then define $m_S^F(A)$ to be the number of composition factors of F which are isomorphic to S, and we define the length $l_F(A)$ to be $\sum m_S^F(A)$, where the sum is taken over all the simple Λ-modules. We define the **length** $l(A)$ of A to be the minimum of $l_F(A)$ for composition series F of A, and $m_S(A)$ to be the minimum of the $m_S^F(A)$. Note that $l(0) = 0$. Our aim is to prove that the numbers $m_S^F(A)$ and $l_F(A)$ are independent of the choice of composition series F.

Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence, and let F be a generalized composition series $B = B_0 \supset B_1 \supset \cdots \supset B_n = 0$ of B. This filtration induces filtrations F' of A given by $A = f^{-1}(B) \supset f^{-1}(B_1) \supset \cdots \supset f^{-1}(B_n) = 0$ and F'' of C given by $C = g(B) \supset g(B_1) \supset \cdots \supset g(B_n) = 0$. We write $f^{-1}(B_i) = A_i$ and $g(B_i) = C_i$. Then we have the following preliminary result.

Proposition 1.1 *Let the notation be as above.*

(a) *The filtrations F' of A and F'' of C are generalized composition series.*

(b) *For each simple module S we have*

$$m_S^{F'}(A) + m_S^{F''}(C) = m_S^F(B).$$

(c) $l_{F'}(A) + l_{F''}(C) = l_F(B)$.

Proof For each $i = 0, \ldots, n$ we have an exact sequence $0 \to A_i \to B_i \to$

$C_i \to 0$ and for each $i = 0, \ldots, n-1$ an exact commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A_{i+1} & \longrightarrow & B_{i+1} & \longrightarrow & C_{i+1} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A_i & \longrightarrow & B_i & \longrightarrow & C_i & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A_i/A_{i+1} & \longrightarrow & B_i/B_{i+1} & \longrightarrow & C_i/C_{i+1} & \to & 0\,. \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Hence we have that if $B_i/B_{i+1} = 0$, then $A_i/A_{i+1} = 0 = C_i/C_{i+1}$. If B_i/B_{i+1} is simple, then either $A_i/A_{i+1} \simeq B_i/B_{i+1}$ and $C_i/C_{i+1} = 0$, or $B_i/B_{i+1} \simeq C_i/C_{i+1}$ and $A_i/A_{i+1} = 0$. Parts (a), (b) and (c) now follow easily. □

We can now prove the Jordan–Hölder theorem.

Theorem 1.2 *Let B be a Λ-module of finite length, and F and G two composition series for B. Then for each simple Λ-module S we have $m_S^F(B) = m_S^G(B) = m_S(B)$, and hence $l_F(B) = l_G(B) = l(B)$.*

Proof We prove this by induction on $l(B)$. Our claim clearly holds if $l(B) \leq 1$. Assume now that $l(B) > 1$. Then B contains a nonzero submodule $A \neq B$. Since it follows by Proposition 1.1 that $l(A) + l(B/A) \leq l(B)$, we have $l(A) < l(B)$ and $l(B/A) < l(B)$, using that $l(A)$ and $l(B/A)$ are nonzero. Let F and G be two composition series for B and let F' and G' denote the induced filtrations on A and F'' and G'' the induced ones on $C = B/A$. For each simple Λ-module S we have by induction that $m_S^{F'}(A) = m_S^{G'}(A)$ and $m_S^{F''}(C) = m_S^{G''}(C)$. Since $m_S^F(B) = m_S^{F'}(A) + m_S^{F''}(C)$ and $m_S^G(B) = m_S^{G'}(A) + m_S^{G''}(C)$ by Proposition 1.1, we get $m_S^F(B) = m_S^G(B)$, and hence also $l_F(B) = l_G(B)$. □

Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence. If F' is a generalized composition series $A = A_0 \supset A_1 \supset \ldots \supset A_s = 0$ of A and F'' is a generalized composition series $C = C_0 \supset C_1 \supset \ldots \supset C_t = 0$ of C, then it follows as above that we get a generalized composition series $B = B_0 \supset g^{-1}(C_1) \supset \ldots \supset g^{-1}(C_{t-1}) \supset f(A) \supset f(A_1) \ldots \supset f(A_s) = 0$ of B. Using Proposition 1.1 and Theorem 1.2 we then have the following.

Corollary 1.3 *Let* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *be an exact sequence of* Λ*-modules where* A *and* C *have finite length. Then* B *has finite length and* $l(A) + l(C) = l(B)$. □

A semisimple module of finite length is clearly uniquely determined by its composition factors, but this does not hold in general for modules with composition series. For example if B has finite length, then B and $\coprod m_S(B)S$, where the sum is taken over all nonisomorphic simple Λ-modules S, have the same composition factors. Hence all finitely generated Λ-modules are determined by their composition factors if and only if Λ is a semisimple ring. It is however an interesting question when indecomposable modules are determined by their composition factors, and this will be discussed in Chapters VIII and IX.

Before giving the following useful consequence of Corollary 1.3 we recall that a morphism of modules is called a **monomorphism** if it is a one to one map and an **epimorphism** if it is an onto map.

Proposition 1.4 *Let* A *be a* Λ*-module of finite length and* $f : A \to A$ *a* Λ*-homomorphism. Then the following are equivalent.*

(a) f *is an isomorphism.*
(b) f *is a monomorphism.*
(c) f *is an epimorphism.*

Proof This follows directly from the fact that $l(f(A)) + l(A/f(A)) = l(A)$. □

For a ring Λ we denote by $\operatorname{Mod}\Lambda$ the category of left Λ-modules. A subcategory $\mathscr{C}$ of $\operatorname{Mod}\Lambda$ is **closed under extensions** if B is in $\mathscr{C}$ for any exact sequence $0 \to A \to B \to C \to 0$ with A and C in $\mathscr{C}$.

The following characterization of the category of finite length modules which we denote by $\text{f.l.}\,\Lambda$ is useful.

Proposition 1.5

(a) *The category* $\text{f.l.}\,\Lambda$ *is the smallest subcategory of* $\operatorname{Mod}\Lambda$ *closed under extensions and containing the simple modules.*
(b) *A* Λ*-module* A *is of finite length if and only if* A *is both artin and noetherian.*

Proof (a) The category $\text{f.l.}\,\Lambda$ contains the simple modules, and is closed under extensions by Corollary 1.3. It is also clear that any subcategory of $\operatorname{Mod}\Lambda$ closed under extensions and containing the simple modules must contain $\text{f.l.}\,\Lambda$.

(b) Since artin and noetherian modules are closed under extensions and simple modules are both artin and noetherian, we have that any module of finite length is artin and noetherian.

Suppose now that a module B is both noetherian and artin. Clearly every submodule and every factor module of B have these properties. Since B is noetherian, there exists a submodule A of B maximal with respect to being of finite length. If $A \neq B$, then B/A has a simple submodule C since B/A is artin. Let A' be the submodule of B containing A such that $A'/A \simeq C$. Then A' is of finite length, which contradicts the maximality of A. Hence we get $A = B$ and so B is of finite length. □

For semisimple modules it is easy to see that any one of the chain conditions implies the other one. Hence we have the following.

Proposition 1.6 *For a semisimple Λ-module B the following are equivalent.*

(a) *B has finite length.*
(b) *B is noetherian.*
(c) *B is artin.* □

A useful point of view concerning the composition factors of a module of finite length is to study a special group associated with the finite length modules. Since the finite length modules are finitely generated by Proposition 1.5, the collection of isomorphism classes of modules of finite length is a set. Hence we can associate with the category of finite length modules f.l. Λ the free abelian group $F(\text{f.l.}\,\Lambda)$ with basis the isomorphism classes $[A]$ of finite length modules A. Denote by $R(\text{f.l.}\,\Lambda)$ the subgroup of $F(\text{f.l.}\,\Lambda)$ generated by expressions $[A]+[C]-[B]$ for each exact sequence $0 \to A \to B \to C \to 0$ in f.l. Λ. Then the **Grothendieck group** $K_0(\text{f.l.}\,\Lambda)$ of f.l. Λ is defined to be the factor group $F(\text{f.l.}\,\Lambda)/R(\text{f.l.}\,\Lambda)$. Associated with a finite length module A is the coset of the isomorphism class $[A]$ in the Grothendieck group $K_0(\text{f.l.}\,\Lambda)$, which we also denote by $[A]$. It turns out that this element $[A]$ in $K_0(\text{f.l.}\,\Lambda)$ contains all information on the composition factors of A. It follows directly that $K_0(\text{f.l.}\,\Lambda)$ is generated by elements $[S]$ where S is a simple Λ-module. Using the Jordan–Hölder theorem we get the following stronger result.

Theorem 1.7 *$K_0(\text{f.l.}\,\Lambda)$ is a free abelian group with basis $\{[S_i]\}_{i\in I}$, where the S_i are the simple Λ-modules and for each finite length module A we have that $[A] = \sum_{i\in I} m_{S_i}(A)[S_i]$ in $K_0(\text{f.l.}\,\Lambda)$.*

Proof Let $F(\text{s.s.}\,\Lambda)$ be the subgroup of $F(\text{f.l.}\,\Lambda)$ generated by the $[S_i]$, where the S_i are a complete set of simple Λ-modules up to isomorphism. Define $\alpha\colon F(\text{s.s.}\,\Lambda) \to K_0(\text{f.l.}\,\Lambda)$ by $\alpha([S_i]) = [S_i]$ in $K_0(\text{f.l.}\,\Lambda)$. We have seen that if $0 \to A \to B \to C \to 0$ is an exact sequence in f.l.$\,\Lambda$, then $m_S(B) = m_S(A) + m_S(C)$ for all simple Λ-modules S. Therefore if for each A in f.l.$\,\Lambda$ we let $\beta([A])$ be the element $\sum_{i\in I} m_{S_i}(A)[S_i]$ in $F(\text{s.s.}\,\Lambda)$, we obtain a morphism $\beta\colon K_0(\text{f.l.}\,\Lambda) \to F(\text{s.s.}\,\Lambda)$. It is now not difficult to see that $\beta\alpha = 1_{F(\text{s.s.}\,\Lambda)}$ and $\alpha\beta = 1_{K_0(\text{f.l.}\,\Lambda)}$. Hence $\alpha\colon F(\text{s.s.}\,\Lambda) \to K_0(\text{f.l.}\,\Lambda)$ is an isomorphism, giving our desired result. □

2 Right and left minimal morphisms

In this section we introduce the concepts of right minimal and left minimal morphisms between modules. These notions are especially interesting for modules of finite length, and they also specialize to the concepts of projective covers and injective envelopes.

Let Λ be an arbitrary ring. For a fixed Λ-module C, consider the category $\operatorname{Mod}\Lambda/C$ whose objects are the Λ-morphisms $f\colon B \to C$, and where a morphism $g\colon f \to f'$ from $f\colon B \to C$ to $f'\colon B' \to C$ is a Λ-morphism $g\colon B \to B'$ such that

$$\begin{array}{ccc} B & \xrightarrow{f} & C \\ \downarrow{\scriptstyle g} & \nearrow_{f'} & \\ B' & & \end{array}$$

commutes. It follows that $g\colon f \to f'$ is an isomorphism in $\operatorname{Mod}\Lambda/C$ if and only if the associated morphism $g\colon B \to B'$ is an isomorphism in $\operatorname{Mod}\Lambda$. We say that $f\colon B \to C$ is **right minimal** if every morphism $g\colon f \to f$ is an automorphism. We introduce an equivalence relation on the objects of $\operatorname{Mod}\Lambda/C$ by defining $f \sim f'$ if $\operatorname{Hom}(f, f') \neq \emptyset$ and $\operatorname{Hom}(f', f) \neq \emptyset$. We now show that for modules of finite length, each equivalence class contains a right minimal morphism.

Proposition 2.1 *Let Λ be a ring and C a Λ-module. Every equivalence class in $\operatorname{Mod}\Lambda/C$ containing some $f\colon B \to C$ with B of finite length contains a right minimal morphism, which is unique up to isomorphism.*

Proof Choose $f\colon B \to C$ in the given equivalence class with $l(B)$ smallest possible, and let $g\colon f \to f$ be a morphism in $\operatorname{Mod}\Lambda/C$. We then have a commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{f} & C \\ \downarrow & \nearrow{\scriptstyle f|_{g(B)}} & \uparrow{\scriptstyle f} \\ g(B) & \hookrightarrow & B \end{array}$$

which shows that $g(B) = B$ by minimality of $l(B)$. Then $g : f \to f$ must be an isomorphism, so that $f : B \to C$ is right minimal.

Assume that $f' : B' \to C$ is a right minimal morphism which is equivalent to $f : B \to C$. We then have morphisms $g : f \to f'$ and $h : f' \to f$. Using that both f and f' are right minimal, we get that hg and gh are isomorphisms. Hence h and g are isomorphisms. □

Let $f : B \to C$ be a morphism with B of finite length. Then the unique, up to isomorphism in $\operatorname{Mod} \Lambda/C$, right minimal morphisms in the equivalence class in $\operatorname{Mod} \Lambda/C$ of f are called **right minimal versions of** f.

Whenever there is a morphism of Λ-modules $f : M \to N$ and M' is a submodule of M then $f|_{M'} : M' \to N$ denotes the restriction of f to M'. The next result gives a reduction to right minimal morphisms.

Theorem 2.2 *Let Λ be a ring and C a Λ-module. Let $g : X \to C$ be an object in* $\operatorname{Mod} \Lambda/C$ *with X of finite length. Then there is a decomposition $X = X' \coprod X''$ such that $g|_{X'} : X' \to C$ is right minimal and $g|_{X''} = 0$. Moreover, the morphism $g|_{X'}$ is a right minimal version of g.*

Proof Choose $f : B \to C$ minimal and equivalent to $g : X \to C$, as is possible by Proposition 2.1. We then have a commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{f} & C \\ \downarrow{\scriptstyle s} & & \| \\ X & \xrightarrow{g} & C \\ \downarrow{\scriptstyle t} & & \| \\ B & \xrightarrow{f} & C\,. \end{array}$$

Then $f = fts$, so that ts is an isomorphism. Letting $\operatorname{Im} s$ denote the image of s and $\operatorname{Ker} t$ the kernel of t, we get $X = \operatorname{Im} s \coprod \operatorname{Ker} t$, and $g|_{\operatorname{Im} s}$ is right minimal and $g|_{\operatorname{Ker} t} = 0$. It is easy to see that $g|_{\operatorname{Im} s}$ is in the same equivalence class as g in $\operatorname{Mod} \Lambda/C$. □

We state the following easy consequence.

Corollary 2.3 *The following are equivalent for a morphism $f: B \to C$ with B of finite length.*

(a) *f is right minimal.*
(b) *If B' is a nonzero summand of B, then $f|_{B'} \neq 0$.* □

For a fixed Λ-module A consider the category $\operatorname{Mod} \Lambda \backslash A$ whose objects are the Λ-morphisms $f: A \to B$ and where a morphism $g: f \to f'$ from $f: A \to B$ to $f': A \to B'$ is a Λ-morphism $g: B \to B'$ such that $gf = f'$. Dual to the notion of right minimal morphism we define a morphism $f: A \to B$ of Λ-modules to be **left minimal** if whenever $g: B \to B$ has the property that $gf = f$, then g is an automorphism. We also have the following dual version of Proposition 2.1. In each equivalence class in $\operatorname{Mod} \Lambda \backslash A$ of a morphism $h: A \to Y$ with Y of finite length there are unique, up to isomorphism in $\operatorname{Mod} \Lambda \backslash A$, left minimal morphisms called the **left minimal versions** of h. For the convenience of the reader we state the following dual result.

Theorem 2.4 *Let Λ be a ring and A a Λ-module. Let $f: A \to Y$ be an object in* $\operatorname{Mod} \Lambda \backslash A$ *with Y of finite length. Then there is a decomposition $Y = Y' \coprod Y''$ such that $p'f: A \to Y'$ is left minimal and $p''f: A \to Y''$ is zero, where $p': Y \to Y'$ and $p'': Y \to Y''$ are the projections according to the decomposition $Y = Y' \coprod Y''$. Moreover, $p'f$ is a left minimal version of f.* □

3 Radical of rings and modules

In this section we give the definition and basic properties of the radical of rings and modules for left artin rings. We are mainly interested in rings Λ where all finitely generated Λ-modules have finite length. This clearly holds for semisimple rings. Actually, we prove that Λ has this property if and only if the Λ-module Λ has finite length, or equivalently, Λ is both left noetherian and left artin. We show that it is superfluous to assume left noetherian.

First we recall that the **radical** of a ring Λ, which we denote by $\mathfrak{r}_\Lambda$, or simply $\mathfrak{r}$, is the intersection of the maximal left ideals of Λ, as well as the intersection of the maximal right ideals of Λ, and is hence an ideal, where an ideal of Λ always means a two-sided ideal. The radical plays a central role in the theory of left artin rings. We recall Nakayama's lemma which states that a left ideal $\mathfrak{a}$ is contained in $\mathfrak{r}$ if and only if $\mathfrak{a}M = M$ implies $M = 0$ when M is a finitely generated Λ-module.

We now prove that left artin rings are left noetherian.

Proposition 3.1 *Assume that Λ is a left artin ring. Then we have the following.*

(a) *The radical $\mathfrak{r}$ of Λ is nilpotent.*
(b) *$\Lambda/\mathfrak{r}$ is a semisimple ring.*
(c) *A Λ-module A is semisimple if and only if $\mathfrak{r}A = 0$.*
(d) *There is only a finite number of nonisomorphic simple Λ-modules.*
(e) *Λ is left noetherian.*

Proof (a) Since Λ is left artin and $\Lambda \supset \mathfrak{r} \supset \mathfrak{r}^2 \supset \cdots \supset \mathfrak{r}^n \supset \cdots$ is a descending sequence of left ideals, there is some n such that $\mathfrak{r}^n = \mathfrak{r}^{n+1}$. Suppose $\mathfrak{r}^n \neq 0$. Then $\mathfrak{r}^{n+1} = \mathfrak{r}^n\mathfrak{r} = \mathfrak{r}^n \neq 0$, so the class $\mathscr{F}$ of all left ideals $\mathfrak{a}$ with $\mathfrak{r}^n\mathfrak{a} \neq 0$ is nonempty. Choose a left ideal $\mathfrak{a}$ in Λ which is minimal in $\mathscr{F}$. Then there is some x in $\mathfrak{a}$ with $\mathfrak{r}^n x \neq 0$ and therefore $\mathfrak{r}^n(\Lambda x) \neq 0$. By the minimality of $\mathfrak{a}$ we have $\mathfrak{a} = \Lambda x$, so $\mathfrak{a}$ is a finitely generated left ideal. Now $0 \neq \mathfrak{r}^n\mathfrak{a} = \mathfrak{r}^{n+1}\mathfrak{a} = \mathfrak{r}^n\mathfrak{r}\mathfrak{a}$, so $\mathfrak{r}\mathfrak{a}$ is also in $\mathscr{F}$ and therefore $\mathfrak{a} = \mathfrak{r}\mathfrak{a}$. This is a contradiction by Nakayama's lemma, and hence $\mathfrak{r}^n = 0$.

(b) Let I be an ideal in Λ containing $\mathfrak{r}$ such that $I/\mathfrak{r}$ is nilpotent in $\Lambda/\mathfrak{r}$. Then there is an integer t with $I^t \subset \mathfrak{r}$. Since $\mathfrak{r}^n = 0$, we have $I^s = 0$ for $s = nt$. Let $\mathfrak{m}$ be a maximal left ideal in Λ, and consider the natural map $p: \Lambda \to \Lambda/\mathfrak{m}$. If $I \not\subset \mathfrak{m}$, then $p(I) \neq 0$, and hence $p(I) = \Lambda/\mathfrak{m}$ since $\Lambda/\mathfrak{m}$ is a simple Λ-module. Then we get $p(I^2) = Ip(I) = I(\Lambda/\mathfrak{m}) = \Lambda/\mathfrak{m}$, and further $0 = p(I^s) = \Lambda/\mathfrak{m}$, a contradiction. This shows that $I \subset \mathfrak{m}$, and hence $I \subset \mathfrak{r}$, so that $I/\mathfrak{r}$ is 0 in $\Lambda/\mathfrak{r}$. Since $\Lambda/\mathfrak{r}$ has no nonzero nilpotent ideals, and is left artin since Λ is left artin, we conclude that $\Lambda/\mathfrak{r}$ is a semisimple ring.

(c) If for a Λ-module A we have that $\mathfrak{r}A = 0$, then A is a $(\Lambda/\mathfrak{r})$-module and hence a semisimple $(\Lambda/\mathfrak{r})$-module. Hence A is also a semisimple Λ-module.

Conversely it is clear by the definition of $\mathfrak{r}$ that if A is a semisimple Λ-module, then $\mathfrak{r}A = 0$.

(d) Since there is only a finite number of nonisomorphic simple $(\Lambda/\mathfrak{r})$-modules and every simple Λ-module is a $(\Lambda/\mathfrak{r})$-module, there is only a finite number of nonisomorphic simple Λ-modules.

(e) It follows from (a) that Λ has a finite filtration $\Lambda \supset \mathfrak{r} \supset \mathfrak{r}^2 \supset \cdots \supset \mathfrak{r}^n = 0$. We write $\Lambda = \mathfrak{r}^0$. Each $\mathfrak{r}^i/\mathfrak{r}^{i+1}$ is a semisimple Λ-module by (c) for $i = 0, 1, \ldots, n-1$ and is artin since Λ is a left artin ring. Hence $\mathfrak{r}^i/\mathfrak{r}^{i+1}$

is noetherian by Proposition 1.6, and consequently Λ is a left noetherian ring. □

We now have the following description of the rings where all finitely generated modules have finite length.

Corollary 3.2 *For a ring Λ the following are equivalent.*

(a) *Every finitely generated Λ-module has finite length.*
(b) *Λ is left artin.*
(c) *The radical $\mathfrak{r}$ of Λ is nilpotent and $\mathfrak{r}^i/\mathfrak{r}^{i+1}$ is a finitely generated semisimple module for all $i \geq 0$.* □

In general it may be difficult to compute the radical of a left artin ring Λ by first finding the maximal left ideals. The following criterion is usually easy to apply. However, before giving this result, it is convenient to introduce the following notation. If A and B are submodules of a module C, we denote by $A + B$ the submodule of C generated by A and B.

Proposition 3.3 *Let Λ be a left artin ring and $\mathfrak{a}$ an ideal in Λ such that $\mathfrak{a}$ is nilpotent and $\Lambda/\mathfrak{a}$ is semisimple. Then we have $\mathfrak{a} = \mathfrak{r}$.*

Proof Let $\mathfrak{a}$ be a nilpotent ideal with $\Lambda/\mathfrak{a}$ semisimple. To show that $\mathfrak{a} \subset \mathfrak{r}$, assume to the contrary that there is a maximal ideal $\mathfrak{m}$ in Λ with $\mathfrak{a} \not\subset \mathfrak{m}$. Then $\mathfrak{a} + \mathfrak{m} = \Lambda$, where $\mathfrak{a} + \mathfrak{m}$ denotes the smallest left ideal containing $\mathfrak{a}$ and $\mathfrak{m}$. Hence $\mathfrak{a}^2 + \mathfrak{a}\,\mathfrak{m} = \mathfrak{a}$, so that $\mathfrak{a}^2 + \mathfrak{m} = \Lambda$. Continuing this way, we get $\mathfrak{a}^n + \mathfrak{m} = \Lambda$ for all n, which gives a contradiction since $\mathfrak{a}$ is nilpotent. This shows $\mathfrak{a} \subset \mathfrak{m}$, and consequently $\mathfrak{a} \subset \mathfrak{r}$.

Clearly the radical of $\Lambda/\mathfrak{a}$ is equal to $\mathfrak{r}/\mathfrak{a}$, so that $\mathfrak{r} = \mathfrak{a}$ since $\Lambda/\mathfrak{a}$ is semisimple. □

For left artin rings the radical of a module also plays an important role. The **radical** $\operatorname{rad} A$ of a Λ-module A over an arbitrary ring Λ is the intersection of the maximal submodules. We have the following useful characterization of the radical of a module. Recall first that a submodule B of a Λ-module A is **small** in A if $B + X = A$ for a submodule X of A implies $X = A$.

Lemma 3.4 *Let A be a finitely generated module over an arbitrary ring Λ. Then a submodule B of A is small in A if and only if $B \subset \operatorname{rad} A$.*