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p-Automorphisms of Finite *p*-Groups

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Contents

Preface

This is a compilation of the lectures given in 1990–97 in the universities of Novosibirsk, Freiburg (in Breisgau), Trento and Cardiff. The book gives a concise account of several, mostly very recent, theorems on the structure of finite p-groups admitting p-automorphisms with few fixed points. The proofs, given in full detail, require various powerful methods of studying nilpotent p-groups; these methods are presented in the manner of a textbook, accessible for students with only a basic knowledge of linear algebra and group theory. Every chapter ends with exercises which vary from elementary checks to relevant results from research papers (but none of them is referred to in the proofs).

By the classical theorems of G. Higman, V. A. Kreknin and A. I. Kostrikin, a Lie ring is soluble (nilpotent) if it has a fixed-point-free automorphism of finite (prime) order. (These Lie ring theorems are also included along with all necessary preliminary material.) Prompted by and based on these Lie ring results, the main theorems of the book state that a finite p-group is close to being soluble (nilpotent) in terms of the order of a p-automorphism and the number of its fixed points. These results can be viewed as general structure theorems about finite p-groups. They are closely related to the theory of (pro-) p-groups of maximal class and given coclass and have natural extensions to locally finite p-groups.

Presenting linear (mostly Lie ring) methods in the theory of nilpotent groups is another main objective of the book. Of course, the methods are judged as tools yielding certain results; on the other hand, the results themselves can be viewed as an excuse for presenting the methods. The proofs of the main results involve viewing automorphisms as linear transformations, associated Lie rings, theory of powerful p-groups, the correspondences of A. I. Mal'cev and M. Lazard given by the Baker–Hausdorff Formula. Applications of the Baker–Hausdorff Formula are rare in the theory of finite p-groups; remarkably, the Mal'cev Correspondence is an essential ingredient in the proof of one of the main results, and the Lazard Correspondence is used to make easier reductions to Lie rings in the proofs of two others.

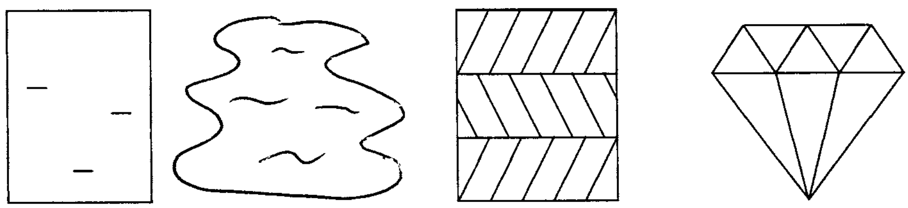
During the preparation of this book the author enjoyed the hospitality of the School of Mathematics of the University of Wales, Cardiff, being a visiting research professor there, and, at earlier stages, of the Department of Mathematics of Trento University, as a *Professore a contratto*. The author is grateful to his colleagues, whose attention and advice helped a lot to improve the presentation; in particular, the author thanks A. Caranti, O. H. Kegel, J. C. Lennox, N. Yu. Makarenko, V. D. Mazurov, Yu. A. Medvedev, A. Shalev, and J. Wiegold.

Introduction

Many problems in group theory arise from the fact that the group operation may not be commutative: in general $ab \neq ba$, for elements a, b of a group. It is natural to distinguish classes of groups with respect to how close they are to commutative (abelian) ones. To measure the deviation from commutativity the *commutator* of the elements a, b is defined to be $[a, b] = a^{-1}b^{-1}ab$. It is easy to see that $ab = ba$ if and only if $[a, b] = 1$ (we use 1 to denote the neutral element of a group). Now, a group G is abelian if and only if $[x, y] = 1$ for all $x, y \in G$ (in other words, if the law $[x, y] = 1$ holds on G).

Iterated commutators give rise to generalizations of abelian groups, other classes of groups that are less commutative, although, in a way, close to commutative ones. So $[...[[x_1, x_2], x_3], \dots, x_k]$ is a *simple* (or *left-normed*) commutator of weight k in the elements $x_1, x_2, \dots, x_k$. A group is said to be *nilpotent of nilpotency class* $\leq c$ if $[...[[x_1, x_2], x_3], \dots, x_{c+1}] = 1$ for any elements $x_1, x_2, \dots, x_{c+1}$. Another way of taking iterated commutators defines the class of soluble groups of derived length $\leq d$. These are groups satisfying the law $\delta_d(x_1, x_2, \dots, x_{2^d}) = 1$, where, recursively, $\delta_1(x_1, x_2) = [x_1, x_2]$, and $\delta_{k+1}(x_1, x_2, \dots, x_{2^{k+1}}) = [\delta_k(x_1, x_2, \dots, x_{2^k}), \delta_k(x_{2^k+1}, x_{2^k+2}, \dots, x_{2^{k+1}})]$. These generalizations (and many others) can also be defined via existence of normal series with commutative or central factors.

The more commutative is the group operation, the friendlier seems the group. For example, the finite abelian groups admit a well-known description. On the other hand, each area of mathematics has its own problems, so does the theory of abelian groups (even the theory of finite abelian groups contains, in a way, a large portion of number theory with its difficult problems). In fact, many areas of mathematics are studied modulo others, more transparent from some viewpoint. Having commutativity in mind, we can build up the following kind of series of classes of finite groups, in the order of decreasing commutativity of the group operation:



abelian $\geq$ nilpotent $\geq$ soluble $\geq \dots \geq$ simple

(Extending this picture to arbitrary infinite groups, we could place polynilpotent and polycyclic between nilpotent and soluble, or insert other generalizations of solubility to the right of soluble; free groups, which do not satisfy any non-trivial laws, could be placed somewhere on the right, etc.)

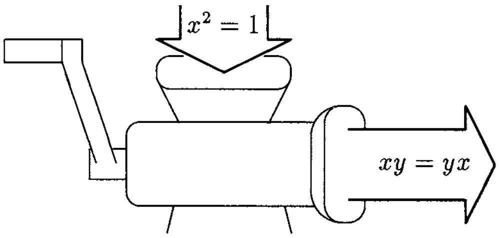
We shall be dealing mostly with nilpotent groups in these lectures. The picture helps to describe a vague feeling about the place which the class of nilpotent groups occupies in finite group theory. Abelian groups seem to be the most transparent, admitting the well-known classification. The (non-abelian) simple groups are, probably, the most non-commutative finite groups: they have no proper normal subgroups, in contrast with abelian groups, all of whose subgroups are normal, or with nilpotent or soluble groups, which have plenty of normal subgroups. Nevertheless, the finite simple groups are also being classified, although this classification is very difficult, occupies thousands of pages of research articles, and is one of the main achievements of mathematics in this century. The finite simple groups seem to have very rigid structure, like crystals; they can be reconstructed from a small fragment (the centralizer of an involution, say). The above remark about studying modulo something is fully applied here: a typical result on non-soluble finite groups is a statement about the factor-group over the largest normal soluble subgroup, or a statement characterizing the (non-abelian) chief factors of a group. The structure of soluble groups appears in certain layers, nilpotent or abelian sections. A typical result about finite soluble groups is bounding the length of the shortest normal series with nilpotent factors, while the structure of these nilpotent factors remains unknown, but is considered good enough, so to say. The difference of soluble finite groups from simple ones seems similar to the difference of graphite from diamond.

The class of nilpotent groups is more like a marsh, a swamp, because their structure appears to be rather amorphous and because of their notorious diversity. There are actually special works showing that there are lots and lots of nilpotent groups and that a classification of nilpotent groups is, in a way, impossible. This is why it is important to have certain beacon lights, marks, showing the ways in this swamp, giving an idea where to work and what kind of results are good in the theory of nilpotent groups.

From the above "more or less commutativity" viewpoint, it is clear that a result in group theory is the better, the more commutativity it yields at the output. For example, one of the "beacon lights" is the Burnside Problem, which can be viewed as asking whether the identity $x^n = 1$ implies some kind of commutativity of the group operation.

Example. Suppose that $x^2 = 1$ for every element x in a group G. Then G is abelian.

Proof. For any two elements $a, b \in G$ we have $abab = 1$. Multiplying on the left by a and then by b we get $baabab = ba$. On the left-hand side we have $baabab = b(a^2)bab = b^2ab = ab$. As a result, $ab = ba$ for any a and b, as required.



For sufficiently large exponents, there exist "bad" groups that are very far from being commutative (groups of S. I. Adyan and P. S. Novikov and of A. Yu. Ol'shanskii). However, if we restrict attention to finite groups only (which means considering the Restricted Burnside Problem), then the results are positive. Using the W. Magnus – A. N. Sanov reduction to Lie algebras, A. I. Kostrikin proved that a d-generator finite group of prime exponent p is nilpotent of class bounded by a function depending on p and d. (Yu. P. Razmyslov showed that the nilpotency class can increase unboundedly with the growth of d, for $p \geq 5$.) E. I. Zelmanov proved that a d-generator finite group of a prime-power exponent p^k is nilpotent of class bounded by a function of p^k and d. Together with the Reduction Theorem of P. Hall and G. Higman, this completes the positive solution of the Restricted Burnside Problem for all exponents in the class of soluble groups and, modulo the classification of the finite simple groups, for all finite groups: for every pair of natural numbers d and n, there exist only finitely many d-generator finite groups of exponent n.

Apart from groups of given exponent, there are other ways of choosing interesting classes of nilpotent groups. For example, N. Blackburn introduced p-groups of maximal class, that is, groups of order p^n and nilpotency class $n - 1$ (which is maximal possible for this order). These groups became a starting point for the theory of p-groups and pro-p-groups of given coclass (S. Donkin, C. R. Leedham-Green, A. Mann, S. McKay, M. Newman, W. Plesken, A. Shalev, E. I. Zelmanov and others). Another interesting generalization are the so-called thin p-groups (and pro-p-groups) where substantial progress was recently made by R. Brandl, A. Caranti, S. Mattarei, M. Newman and C. Scoppola.

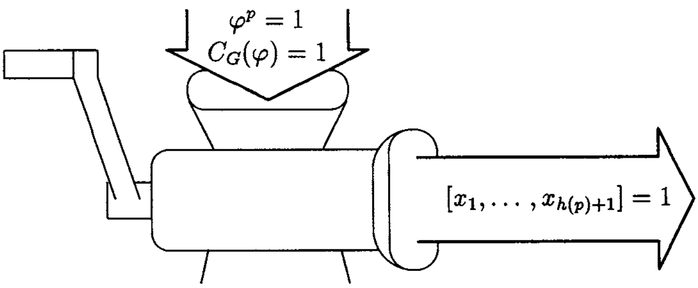
In these lectures we shall consider nilpotent p-groups admitting certain p-automorphisms. A bijection φ of a group G onto itself is an *automorphism* if it preserves the group operation: $(xy)^\varphi = x^\varphi y^\varphi$ for all $x, y \in G$. For any $g \in G$, one can form the *inner automorphism* $\tau_g : x \to g^{-1}xg$. It is easy to see that τ_g is the identity mapping of G if and only if g commutes with all elements of G. So, modulo commutative groups, studying a group is equivalent to the study of its (inner) automorphisms. Conversely, all automorphisms of a group can be regarded as inner automorphisms of a larger group. Thus, studying automorphisms of groups is, in a way, equivalent to studying groups themselves; this approach sometimes provides certain advantages. For example, the theory of p-groups of maximal class is virtually equivalent to that of p-groups admitting an automorphism of order p with exactly p fixed points, and a large portion of the theory of p-groups of given coclass amounts to that of p-groups admitting an automorphism of order p^k with exactly p fixed points.

Again, results on automorphisms may be considered the better, the more commutativity they yield.

Example. Suppose that φ is an automorphism of a finite group G such that $\varphi^2 = 1$ and 1 is the only element of G left fixed by φ (in other words, $x^\varphi \neq x$ whenever $x \neq 1$). Then G is abelian.

Proof. We show first that $G = \{g^{-1}g^\varphi \mid g \in G\}$. Since G is finite, it is sufficient to show that the mapping $g \to g^{-1}g^\varphi$ is injective. If $g_1^{-1}g_1^\varphi = g_2^{-1}g_2^\varphi$, then $g_2g_1^{-1} = g_2^\varphi(g_1^\varphi)^{-1} = (g_2g_1^{-1})^\varphi$, whence $g_2g_1^{-1} = 1$ by hypothesis, that is, $g_2 = g_1$. Now for each $h \in G$ we have $h = g^{-1}g^\varphi$ for some $g \in G$; then $hh^\varphi = g^{-1}g^\varphi(g^{-1}g^\varphi)^\varphi = g^{-1}g^\varphi(g^\varphi)^{-1}g^{\varphi^2} = g^{-1}g = 1$, since $\varphi^2 = 1$. In other words, $h^\varphi = h^{-1}$ for every $h \in G$. Finally, for any $a, b \in G$, we have $ba = (a^{-1}b^{-1})^\varphi = (a^{-1})^\varphi(b^{-1})^\varphi = ab$. □

An automorphism that fixes only the identity element of a group is called *regular*. One can prove that if a finite group admits a regular automorphism of order 3, then it is nilpotent of class at most 2. By a theorem of J. G. Thompson, 1959, every finite group with a regular automorphism of prime order is nilpotent. G. Higman in 1957 proved that the nilpotency class of a nilpotent group with a regular automorphism of prime order p is bounded by a function $h(p)$, depending on p only. (V. A. Kreknin and A. I. Kostrikin in 1963 found a new proof giving an explicit upper bound for this Higman's function.) The theorem of J. G. Thompson is an example of a result modulo the theory of nilpotent groups, while the theorem of G. Higman deals with nilpotent groups from the outset.



In fact, the works of G. Higman, V. A. Kreknin and A. I. Kostrikin are essentially about Lie rings with regular automorphisms: the group-theoretic corollaries are rather straightforward consequences. V. A. Kreknin in 1963 also proved that a Lie ring with a regular automorphism of arbitrary finite order n is soluble, of derived length bounded in terms of n. (However, it is still an open problem to obtain an analogous result for nilpotent groups.) These Lie ring results turned out to be useful in the study of p-automorphisms of finite p-groups with few fixed points ("almost regular" ones), although a p-automorphism of a finite p-group can never be regular.

The results that we are aiming at in these lectures are about the structure of a finite p-group P admitting a p-automorphism φ of order p^n with exactly p^m fixed points. First consider the case where $n = 1$, that is, $|\varphi| = p$.

- Then the derived length of P is (p, m)-bounded, that is, bounded in terms of p and m only [J. Alperin, 1962].
- Moreover, in a match to G. Higman's Theorem, P contains a subgroup of (p, m)-bounded index which is nilpotent of p-bounded class [E. I. Khukhro, 1985].
- In another direction, prompted by the results on p-groups of maximal class, P also contains a subgroup of (p, m)-bounded index which is nilpotent of m-bounded class [Yu. A. Medvedev, 1994a,b].

Now consider the general situation, with $|\varphi| = p^n$.

- Then the derived length of P is bounded in terms of p, n and m [A. Shalev, 1993a].
- Moreover, in a match to Kreknin's Theorem, P contains a subgroup of (p, n, m)-bounded index which is soluble of p^n-bounded derived length [E. I. Khukhro, 1993a].
- In the extreme case of $m = 1$, where the number of fixed points is p, minimal possible, P contains a subgroup of (p, n)-bounded index which is nilpotent of class 2 (for $|\varphi| = p$, [R. Shepherd, 1971], and [C. R. Leedham-Green and S. McKay, 1976]; for $|\varphi| = p^n$, [S. McKay, 1987], and [I. Kiming, 1988]).

(Most of the above results can be extended to certain classes of infinite groups, but we do not discuss these generalizations in the book.)

The "modular" case, where a p-automorphism acts on a p-group, turned out to be easier than the "ordinary" one, where the order of the automorphism is coprime to the order of the group. It is still an open problem to obtain an analogue of Kreknin's Theorem for nilpotent groups with regular automorphisms of finite order. The only known cases are those of prime order [G. Higman, 1957] and order four [L. G. Kovács, 1961]. There is also a generalization of Higman's Theorem for nilpotent groups with an almost regular automorphism of prime order ([E. I. Khukhro, 1990] and [Yu. Medvedev, 1994c]); see also the book [E. I. Khukhro, 1993b].

Applications of Lie rings and other linear tools are based on the fact that nilpotent groups are close to commutative ones. Abelian subgroups and sections are similar to vector spaces (or modules), and the action of automorphisms on such sections is similar to linear transformations. The group commutators can be used to define the structure of a Lie ring on the direct sum of the additively written factors of the lower central series of a group, the so-called associated Lie ring. Other ways of constructing a Lie ring from a nilpotent group, the correspondences of A. I. Mal'cev and M. Lazard, are based on the Baker–Hausdorff Formula. Under these correspondences, the Lie ring reflects the properties of the group in a much better way, but this technique cannot be applied to any nilpotent group.

The Lie ring method of solving group-theoretic problems consists of three major steps. First, the problem must be translated into a corresponding problem about Lie rings constructed from the groups. Then the Lie ring problem is solved. The results on Lie rings must then be translated back, into required conclusions about groups. The advantage lies in the fact that it is usually easier to deal with Lie rings as more linear objects; for example, one can extend the ground ring, which gives rise to the analogues of eigenspaces with respect to the automorphism, etc. On the other hand, both crossings over, from groups to Lie rings and back, may be quite non-trivial. For example, the number of fixed points of an automorphism may well be much greater on the associated Lie ring than on the group. Another example: if the Lie ring result gives a subring of small nilpotency class and small index, say, this does not immediately give the required subgroup in the group, since there is no good correspondence between subrings of the associated Lie ring and subgroups of the group (such kind of difficulty had to be overcome in the theorem on almost regular coprime automorphism in [E. I. Khukhro, 1990]). Thus, reductions to Lie rings and recovering information about the group from the Lie ring results may sometimes require even more effort than the Lie ring theorems themselves.

The difficulty in proving an analogue of Kreknin's Theorem for nilpotent groups with a regular automorphism of composite order lies in the fact that the derived length of the associated Lie ring may be smaller than that of the group. What makes the "modular" case that we are dealing with in this book much more friendly is the bounds for the ranks of all abelian sections that follow from the restrictions on the number of fixed points. This reduces the proofs to powerful p-groups whose nice linear properties make it easier to apply the Lie ring methods.

Lie rings are used in the book not only for proving the main results, but also for deriving many of the standard "linear" properties of nilpotent groups. Naturally, quite a lot of preliminary material on Lie rings is included.

We tried to make the book closer to a textbook, really accessible to students with only undergraduate knowledge in algebra and group theory; efforts were made to ensure that there are no stumbling-blocks disguised by the words "obvious" or "as is well-known". The relatively short chapters follow the pattern of the lecture course. The chapters on methods alternate with chapters on applications, so that the reader could see, as soon as possible, the results that can be achieved by these methods. Exercises included in every chapter vary from elementary ones to relevant results from research papers (but none of them is referred to in the proofs).

Chapter 1 contains preliminaries on groups, rings and modules, and on varieties of algebraic systems. Most of this material may well be not more than a reminder, but we note that varietal arguments, often in terms of groups or Lie rings with additional operations, are essential in the subsequent chapters.

Because of the outstanding role of automorphisms in the book, we devoted a special chapter, Chapter 2, to preliminary material on automorphisms, containing a few folklore elementary lemmas on fixed points.

Chapter 3 combines material on nilpotent and soluble groups. Besides definitions and basic properties, it contains some criteria for soluble groups to be nilpotent and a criterion for a variety to be soluble.

In Chapter 4 elementary properties of finite p-groups are proved, as well as a theorem of P. Hall on the orders of the lower central factors of a normal subgroup.

Chapter 5 introduces Lie rings. A section on soluble and nilpotent Lie rings collects the analogues of group-theoretic results from Chapter 3, including a criterion for a variety to be soluble, which is used in Chapter 7 for proving Kreknin's Theorem. Then free Lie rings are constructed within free associative algebras. The main use of this construction is in Chapters 9 and 10 on the Baker–Hausdorff Formula and the Mal'cev Correspondence; the only fact needed earlier is that free Lie rings are multihomogeneous with respect to free generators.

Chapter 6 introduces one of our main tools, the associated Lie rings; in § 6.3 they are used to derive several properties of nilpotent groups.

Theorems of G. Higman, V. A. Kreknin and A. I. Kostrikin on regular automorphisms of Lie rings are proved in Chapter 7 in generalized combinatorial form: first for $(\mathbb{Z}/n\mathbb{Z})$-graded Lie rings, then for free Lie rings, and finally for Lie rings with automorphisms. Although it is only these combinatorial results that are used later, we could not help deriving the consequences for Lie rings and finite nilpotent groups with regular automorphisms.

Technique accumulated to this point enables us to prove in Chapter 8 the first of the main results, an analogue of G. Higman's Theorem for p-groups with an almost regular automorphism of order p.

In Chapter 9 free nilpotent $\mathbb{Q}$-powered (torsion-free divisible) groups and nilpotent Lie $\mathbb{Q}$-algebras are constructed within associative algebras. We prove the basic property of the Baker–Hausdorff Formula, which links the group and the Lie ring operations.

In Chapter 10 the Baker–Hausdorff Formula is used to establish the Mal'cev Correspondence between nilpotent $\mathbb{Q}$-powered groups and nilpotent Lie $\mathbb{Q}$-algebras. There is also a similar correspondence of M. Lazard for nilpotent p-groups of class $\leq p-1$. Applications of the Baker–Hausdorff Formula are rare in the theory of finite p-groups; remarkably, they are featured in the proofs of the rest of the main results. In Chapter 12 the Mal'cev Correspondence is substantially used to prove the analogue of Kreknin's Theorem for finite p-groups with an almost regular automorphism of order p^n, and in Chapters 13 and 14 the Lazard Correspondence makes it much easier to perform reductions to Lie rings.

Another important method used in Chapters 12, 13 and 14 is the Lubotzky–Mann theory of powerful p-groups, which is developed in Chapter 11. Every p-group of sectional rank r contains a powerful subgroup of (p,r)-bounded index, which is the "more linear part" of the group. So-called uniformly powerful p-groups enjoy even more linear properties similar to those of homocyclic abelian groups. Bounds for the number of fixed points of a p-automorphism imply bounds for the ranks; this is why powerful p-groups appear naturally in the theory of p-groups with almost regular p-automorphisms.

We already mentioned that Chapter 12 contains an analogue of Kreknin's Theorem for finite p-groups with an almost regular automorphism of order p^n, and that the Mal'cev Correspondence is used in the proof. Another ingredient of the proof is calculations in powerful p-groups, using an important Interchanging Lemma of A. Shalev. Kreknin's Theorem is used twice. First we apply it to the associated Lie ring of a powerful p-group, which already leads to a "weak" bound for the derived length depending on both the number of fixed points and the order of the automorphism. Then Kreknin's Theorem is applied via the Mal'cev Correspondence to a free nilpotent group with an automorphism of finite order. The general result obtained allows us to find a required subgroup of bounded index with a "strong" bound for its derived length, depending only on p^n, the order of the automorphism.

Chapter 13 deals with the extreme case where a finite p-group P admits a p-automorphism φ with just p fixed points, the least possible number. Then the result is extremely strong: P has a subgroup of bounded index which is nilpotent of class ≤ 2 (even abelian for $p = 2$). We give a proof which is different from the original proofs of C. R. Leedham-Green, S. McKay and R. Shepherd (for $|\varphi| = p$) and S. McKay and I. Kiming (for $|\varphi| = p^n$). Although with possibly worse bounds for the index of the subgroup, our proof is more Lie ring oriented, making use of Higman's and Kreknin's Theorems, theory of powerful p-groups, and the Lazard Correspondence. After reduction to Lie rings, we prove independently an analogous theorem on Lie rings which is interesting in its own right. There we adopt the approach of Yu. Medvedev, defining a new "lifted" Lie ring multiplication. Anticipated in the works of A. Shalev and E. I. Zelmanov on p-groups and pro-p-groups of given coclass, Yu. Medvedev's construction is remarkably transparent and elementary.

In Chapter 14 we prove that if a finite p-group P admits an automorphism of order p with p^m fixed points, then P has a subgroup of (p,m)-bounded index which is nilpotent of m-bounded class. Recall that in Chapter 8 we prove that P has a subgroup of bounded index which has p-bounded nilpotency class. Neither of these results follows from the other; which conclusion is better depends on which of the parameters p and m is "much less" than the other. The proof is quickly reduced to Lie rings via the Lazard Correspondence. This reduction to Lie rings based on the result of Chapter 8 is easier than in

Chapter 13, since here we are not constrained by the requirement to obtain such a strong bound for the nilpotency class as 2. In fact, the bulk of Chapter 14 is an independent proof of the analogous Lie ring theorem (of Yu. Medvedev). Much of the technique developed in Chapter 13 is used there, including the new lifted Lie products.

The book [E. I. Khukhro, 1993b] also contains the theorems from Chapters 8 and 12, but in a more condensed form. In the present book the proofs are rewritten (inflated) to make them more accessible for beginners, and all of the background material is included. (Among other topics on automorphisms of nilpotent groups in [E. I. Khukhro, 1993b] are splitting p-automorphisms of finite p-groups with applications to the Hughes problem, almost regular coprime automorphisms of nilpotent groups and Lie rings, and some generalizations of the Restricted Burnside Problem to varieties of operator groups.)

Besides the research papers mentioned in the book, we included several general references in the Bibliography; many of the textbooks may be indicated as our sources, especially for the preliminary chapters. The survey [A. Shalev, 1995] on finite p-groups reflects Lie ring methods, almost regular automorphisms, and applications in the theory of pro-p-groups. Almost regular automorphisms in a broader context of locally finite groups were also surveyed in [B. Hartley, 1987]. Linear methods in the theory of (residually) nilpotent groups and pro-p-groups are the subject of the survey [E. I. Zelmanov, 1995].