Chapter 1

Partial Difference Equations

Discrete linearisable Gambier equations

A K Common†, E Hesameddini† and M Musette‡
† Institute of Mathematics and Statistics,
University of Kent,
Canterbury, CT2 7NF, UK.
‡ Dienst Theoretische Natuurkunde,
Vrije Universiteit Brussel,
Pleinlaan 2, B-1050 Brussels, Belgium

Abstract

We propose two candidates for discrete analogues to the nonlinear Ermakov equation. The first discretisation of degree two possesses the two features which characterise its linearisation in the continuum while the second form of degree one is directly linearisable into a third order equation.

We extend our procedure of discretisation, based on an association with a two dimensional conformal mapping, to a nonlinear equation of the Gambier classification linearisable into a fourth order equation.

1 Introduction

In 1906, Gambier [8] reported that the second order nonlinear differential equation

$$zz_{xx} - \frac{p_x}{p} zz_x - \frac{1}{2} z_x^2 + 2qz^2 - cp^2 = 0, \tag{1}$$

where $c \neq 0$ is a constant and p, q two arbitrary functions of x, is linearisable into a third order equation. Making the transformation $z = py$, one obtains for y the following equation

$$yy_{xx} - \frac{1}{2} y_x^2 + 2fy^2 - c = 0, \tag{2}$$

where

$$f(x) = q(x) + \frac{1}{2}\left(\frac{p_{xx}}{p} - \frac{3}{2}\left(\frac{p_x}{p}\right)^2\right), \tag{3}$$

which by derivation, is simply related to the third order linear equation

$$y_{xxx} + 4fy_x + 2f_x y = 0\ . \tag{4}$$

The expression of the general solution of (2) in terms of two solutions (ψ_1, ψ_2) of the second order linear equation

$$\psi_{xx} + f(x)\psi = 0 \tag{5}$$

was previously given by Ermakov [6, 4] in 1880 as

$$y(x) = \alpha\psi_1^2(x) + 2\beta\psi_1(x)\psi_2(x) + \gamma\psi_2^2(x) \tag{6}$$

with $\alpha\gamma - \beta^2 = -cW^{-2}/2$, the constant W representing the Wronskian of the two solutions of (5).

At this time, Appell [2] also found the relation (6) in searching for the link between the solution of the third order linear equation (4) and the square of the general solution of (5).

Moreover, the transformation

$$y(x) = \varphi(x)Y(X(x)), \quad X_x \neq 0 \tag{7}$$

converts the equation (2) to another equation with constant coefficients

$$YY_{XX} - \frac{1}{2}Y_X^2 = \frac{c}{K^2}, \quad K \text{ arbitrary} \tag{8}$$

under the conditions

$$\varphi(x) = \frac{K}{X_x(x)}, \quad \{X; x\} = 2f(x) \tag{9}$$

where $\{X; x\}$ is the Schwarzian derivative of X with respect to x. Therefore, one can represent the solution of (2) either by the formula (6) or by

$$y(x) = \sqrt{-2c}\,\frac{X}{X_x} \tag{10}$$

with X solution of the third order equation

$$\frac{X_{xxx}}{X_x} - \frac{3}{2}\left(\frac{X_{xx}}{X_x}\right)^2 = 2f(x). \tag{11}$$

We concentrate our attention on the equation (2) and notice that one can set, without implying any restriction, the function $p(x)$ equal to a constant.

In §2, we establish a very useful association between this equation and a particular form of a 2-dimensional conformal Riccati system which again reveals the link between the nonlinear equation (2) and the linear equations (4) and (5).

In the first part of §3, we consider the mappings leading to the difference analogues of the conformal Riccati systems which are linearisable. A discrete version of (2) is built from a particular conformal mapping which coincides, in the continuum limit, with its analogous 2-dimensional Riccati system. The resulting equation of degree two is linearisable into a discrete analogue of (4) and its general solution is related to a discrete analogue of (5) by the quadratic expression (6).

In the second part of §3, we obtain a second form of discrete Ermakov equation by using its link with a discrete Schwarzian explicitly invariant for the homographic transformation. Contrary to the expression given in the first part of the present section, this second form corresponds to a scheme of discretisation of first degree. It is also linearisable into a discrete third order equation with continuous limit identical to (4) but a relationship in a compact form with the solutions of a second order linear difference equation has not yet been achieved.

In §4, we extend our procedure of discretisation, based on a two dimensional conformal mapping, to a nonlinear equation of the Gambier classification linearisable into a fourth order equation.

2 Conformal Riccati system

Two main classes of linearisable coupled Riccati systems are ([1]) :

(1) Projective Riccati equations:

They have the matrix form

$$\boldsymbol{\omega}_x = \mathbf{a} + B\boldsymbol{\omega} + \boldsymbol{\omega}(\mathbf{c}, \boldsymbol{\omega}) \tag{12}$$

where the elements of $N \times N$ matrix B and N dimensional vectors $\mathbf{a}, \mathbf{c}$ are functions of the independent variable x.

(2) Conformal Riccati equations:

$$\boldsymbol{\omega}_x = \boldsymbol{\beta} + E\boldsymbol{\omega} + a\boldsymbol{\omega} + \boldsymbol{\omega}(\boldsymbol{\gamma}, \boldsymbol{\omega}) - \tfrac{1}{2}\boldsymbol{\gamma}(\boldsymbol{\omega}, \boldsymbol{\omega}) \tag{13}$$

with $N \times N$ matrix E satisfying

$$E\tilde{I} + \tilde{I}E^T = 0 \tag{14}$$

where $\tilde{I}$ is such that

$$(\boldsymbol{\delta}, \boldsymbol{\alpha}) \equiv \boldsymbol{\delta}^T \tilde{I} \boldsymbol{\alpha} = \delta_1 \alpha_1 + \delta_2 \alpha_2 + \cdots + \cdots + \delta_p \alpha_p - \delta_{p+1} \alpha_{p+1} - \cdots - \delta_N \alpha_N \quad (15)$$

and $1 \leq p \leq N$.

One can show [11] that specific transformations on the dependent and independent variables of the set of equations (13) lead to a standard form which in case $N = 2$, $\tilde{I} \equiv \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon^2 \end{pmatrix}$ with $\varepsilon^2 = \pm 1$, is

$$u_x = A_1 + D_2 uv \quad (16)$$

$$v_x = A_2(x) - \frac{\varepsilon^2}{2} D_2 \left(u^2 - \varepsilon^2 v^2 \right) \quad (17)$$

where A_1, D_2 are constant and $A_2(x)$ a function of x. Eliminating v from the two coupled first order differential equations (16-17), we obtain for $u(x)$

$$u u_{xx} - \frac{3}{2} u_x^2 + 2A_1 u_x - \frac{1}{2} A_1^2 - A_2 D_2 u^2 + \frac{\varepsilon^2}{2} D_2^2 u^4 = 0, \quad (18)$$

or in the variable $y = u^{-1}$

$$y y_{xx} - \frac{1}{2} y_x^2 + 2A_1 y^2 y_x + \frac{1}{2} A_1^2 y^4 + A_2 D_2 y^2 - \frac{\varepsilon^2}{2} D_2^2 = 0. \quad (19)$$

We see that this is equivalent to (2) when

$$A_1 \equiv 0, \quad A_2(x) = \frac{2f(x)}{D_2}, \quad c = \frac{\varepsilon^2 D_2^2}{2}. \quad (20)$$

Let us note that in the variables $w = -(u + v)/2$, $y = u^{-1}$ and for $\varepsilon^2 = -1$ the system (16-17) becomes

$$w_x + D_2 w^2 + \frac{A_1 + A_2(x)}{2} = 0 \quad (21)$$

$$y_x + A_1 y^2 - D_2(2yw + 1) = 0 \quad (22)$$

which corresponds to the system given by Gambier [9] in association with the equation of class XXVII in the particular case $n = 2$.

If $A_1 \equiv 0$, instead of having two Riccati equations in "cascade", the second equation of the set (21-22) is linear and possesses the particular solution $y = \psi_1 \psi_2$ where (ψ_1, ψ_2) are two solutions of the linear equation

$$\psi_{xx} + \frac{D_2 A_2(x)}{2} \psi = 0. \quad (23)$$

with $w = (\mathrm{Log}\psi_1)_{,x}/D_2$ and the Wronskian $W(\psi_1, \psi_2) = D_2$. Hence performing a linear transformation on (ψ_1, ψ_2), one recovers the solution (6) with $\alpha\gamma - \beta^2 = -1/4$.

On the other hand, the standard way to linearise the conformal Riccati system (13) is to first of all convert it to a projective Riccati system of one higher dimension. For the two dimensional case (16-17) we make the definition

$$w \equiv u^2 + \varepsilon^2 v^2 \tag{24}$$

and obtain

$$\begin{aligned} u_x &= A_1 + D_2 uv && (25)\\ v_x &= A_2 - \frac{1}{2}\varepsilon^2 D_2 w + D_2 v^2 && (26)\\ w_x &= 2\varepsilon^2 A_2 v + D_2 wv + 2A_1 u && (27) \end{aligned}$$

which is a 3-dimensional projective Riccati system. This is equivalent to the linear system

$$\begin{pmatrix} \psi_{1,x} \\ \psi_{2,x} \\ \psi_{3,x} \\ \psi_{4,x} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & A_1 \\ 0 & 0 & -\frac{1}{2}\varepsilon^2 D_2 & A_2 \\ 2A_1 & 2\varepsilon^2 A_2 & 0 & 0 \\ 0 & -D_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \tag{28}$$

when $u = \psi_1/\psi_4$, $v = \psi_2/\psi_4$, $w = \psi_3/\psi_4$.

In the particular case $A_1 \equiv 0$, the linear system degenerates in three coupled equations ($\psi_1 \equiv K =$ constant) and the variable ψ_4 with the identification $D_2 A_2(x) = 2f(x)$ satisfies the third order linear equation (4).

3 Discrete forms of the Ermakov equation

There are of course infinitely many ways of constructing nonlinear difference equations which tend to the equation (2) in the continuum. We will discuss two approaches based on the connection with the conformal Riccati system and Schwarz derivative respectively. Firstly, we will obtain a discrete form whose general solution is connected to a linear second order difference equation in the same way as in the continuum.

The conformal Riccati equations in (13) are the infinitesimal counterpart [1] of the discrete conformal transform of the vector $\boldsymbol{\omega}$ in $\mathbb{R}^N$ given by

$$\boldsymbol{\omega} \to e^{\rho}\Lambda \frac{[\boldsymbol{\omega} + \boldsymbol{\gamma}\boldsymbol{\omega}^2]}{[1 + 2(\boldsymbol{\omega}, \boldsymbol{\gamma}) + \boldsymbol{\omega}^2\boldsymbol{\gamma}^2]} + \boldsymbol{\alpha} \tag{29}$$

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where Λ is a general Lorentz transformation and $\boldsymbol{\gamma}, \boldsymbol{\alpha}$ are in $\mathbb{R}^N$ and ρ is a scalar. The most obvious and natural way to discretise (13) is then to consider (29) as a mapping from $\boldsymbol{\omega}(n)$ to $\boldsymbol{\omega}(n+1)$. To obtain the discrete analogy of the equation (2) we take $N = 2$ as in §2 and the standard form of (29) with $\Lambda = I$, $\rho = 0$, $\boldsymbol{\gamma} = \binom{0}{\gamma_2}$, $\boldsymbol{\alpha} = \binom{\alpha_1}{\alpha_2}$. The mapping (29) may then be written in the component form:

$$u(n+1) - \alpha_1(n) = \frac{u(n)}{\{1 + 2\varepsilon^2\gamma_2(n)v(n) + [u^2(n) + \varepsilon^2 v^2(n)]\varepsilon^2\gamma_2^2(n)\}}, \quad (30)$$

$$v(n+1) - \alpha_2(n) = \frac{\{v(n) + \gamma_2(n)[u^2(n) + \varepsilon^2 v^2(n)]\}}{\{1 + 2\varepsilon^2\gamma_2(n)v(n) + [u^2(n) + \varepsilon^2 v^2(n)]\varepsilon^2\gamma_2^2(n)\}}, \quad (31)$$

and one recovers the standard form (16–17) by setting

$$x = nh, \quad u(n) \to u(x), \quad v(n) \to v(x)$$
$$\gamma_2(n) = -\frac{h\varepsilon^2 D_2}{2}, \quad \alpha_2(n) = hA_2(x), \quad \alpha_1(n) = hA_1, \quad (32)$$

and taking the continuum limit $h \to 0$.

To linearise the set (30-31) we make the definition

$$w(n) = u^2(n) + \varepsilon^2 v^2(n) \quad (33)$$
$$u(n) = \frac{\xi_1(n)}{\xi_4(n)}, \quad v(n) = \frac{\xi_2(n)}{\xi_4(n)}, \quad w(n) = \frac{\xi_3(n)}{\xi_4(n)} \quad (34)$$

and obtain the four linear recurrence relations:

$$\begin{pmatrix} \xi_1(n+1) \\ \xi_2(n+1) \\ \xi_3(n+1) \\ \xi_4(n+1) \end{pmatrix} = \mathcal{A}(n) \begin{pmatrix} \xi_1(n) \\ \xi_2(n) \\ \xi_3(n) \\ \xi_4(n) \end{pmatrix} \quad (35)$$

with

$$\mathcal{A} = \begin{pmatrix} 1 & 2\varepsilon^2\alpha_1\gamma_2 & \varepsilon^2\alpha_1\gamma_2^2 & \alpha_1 \\ 0 & 1 + 2\varepsilon^2\alpha_2\gamma_2 & \gamma_2(1 + 2\varepsilon^2\alpha_2\gamma_2) & \alpha_2 \\ 2\alpha_1 & 2\varepsilon^2(\alpha_2 + \gamma_2\alpha_{12}^2) & 1 + \varepsilon^2\gamma_2(2\alpha_2 + \gamma_2\alpha_{12}^2) & \alpha_{12}^2 \\ 0 & 2\varepsilon^2\gamma_2 & \varepsilon^2\gamma_2^2 & 1 \end{pmatrix} \quad (36)$$

and $\alpha_{12}^2 = \alpha_1^2 + \varepsilon^2\alpha_2^2$.

If $\alpha_1(n) \equiv 0$, $\xi_1(n) = K$(constant) and this set degenerates into three linear recurrence relations between $\xi_2(n), \xi_3(n)$ and $\xi_4(n)$. Moreover, with the identification (32) for $\gamma_2(n)$ and $\alpha_2(n)$, it becomes, in the limit $h \to 0$, identical to the third order linear equation (4) for $\xi_4(x) \equiv (Ku(x))^{-1}$.

A discrete form of the equation (2) may then be obtained by eliminating $v(n)$ between the set of two equations (30-31). From the first of these we have a quadratic equation for $v(n)$ and we choose the root

$$v(n) = \frac{\varepsilon^2}{\gamma_2(n)}\left(-1 + \sqrt{\frac{u(n)}{u(n+1)} - \varepsilon^2\gamma_2^2(n)u^2(n)}\right) \tag{37}$$

to get the correct continuum limit. Substituting in the second one with the following identification

$$x = nh, \quad \alpha_2(n) = 2hf(x), \quad \gamma_2(n) = -\frac{h\varepsilon^2}{2}, \quad u(n) = u(x) \tag{38}$$

we obtain for $y(x) = u(x)^{-1}$ the equation :

$$\begin{aligned} y(x+h)\ 2 - h^2 f(x) &= \sqrt{y(x+h)y(x) - \frac{h^2}{4}\varepsilon^2} \\ &+ \sqrt{y(x+2h)y(x+h) - \frac{h^2}{4}\varepsilon^2}. \end{aligned} \tag{39}$$

The square roots may be eliminated by squaring twice. Introducing the compact notation:

$$\overline{\overline{y}} = y(x+2h), \quad \overline{y} = y(x+h), \quad \underline{y} = y(x-h), \quad \underline{\underline{y}} = y(x-2h) \tag{40}$$

also valid for every function of x, we get the equation

$$\left[\overline{\overline{y}} - y - \left(2 - h^2 f(x)\right)^2 \overline{y}\right]^2 - \left(2 - h^2 f(x)\right)^2 \left(4\overline{y}y - \varepsilon^2 h^2\right) = 0. \tag{41}$$

Setting $y(x) = \psi(x)\,\phi(x)$, $A(x) = \pm(2 - h^2 f(x))$, equation (41) becomes

$$\left(\overline{\overline{\psi}}\,\overline{\overline{\phi}} - \psi\phi - A(x)^2\overline{\psi}\,\overline{\phi}\right)^2 - 4A(x)^2\psi\overline{\psi}\phi\overline{\phi} = -h^2\varepsilon^2 A(x)^2. \tag{42}$$

The left hand side of this relation can be identified with a perfect square if

$$\overline{\overline{\psi}}\,\overline{\overline{\phi}} - \psi\phi - A(x)^2\overline{\psi}\,\overline{\phi} = A(x)\left(\overline{\psi}\phi + \overline{\phi}\psi\right) \tag{43}$$

or

$$\overline{\overline{\psi}}\,\overline{\overline{\phi}} = \left(A(x)\overline{\psi} + \psi\right)\left(A(x)\overline{\phi} + \phi\right). \tag{44}$$

Thus, the condition (43) introduced in (42) implies that

$$\left(\overline{\psi}\phi - \overline{\phi}\psi\right) = \pm ih\varepsilon \tag{45}$$

while the condition (44) leads to

$$\overline{\overline{\psi}}\,\overline{\phi} - \overline{\overline{\phi}}\,\overline{\psi} = A(x)\left(k(x) - k(x)^{-1}\right)\overline{\psi}\,\overline{\phi} + k(x)\psi\overline{\phi} - k(x)^{-1}\phi\overline{\psi} \tag{46}$$

where $k(x)$ is an arbitrary function. Those two last relations are compatible if one identifies $k(x)$ with -1. Therefore, the functions ψ and ϕ are two linear independent solutions of the second order difference equation

$$\Psi(x+2h) + A(x)\Psi(x+h) + \Psi(x) = 0 \tag{47}$$

with the Wronskian equal to $\pm ih\varepsilon$. Applying the linear transformation

$$\begin{pmatrix} \psi(x) \\ \phi(x) \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \psi(x) \\ \phi(x) \end{pmatrix}, \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$$

the solution of the difference equation (41) becomes

$$y(x) = \alpha\psi(x)^2 + 2\beta\psi(x)\,\phi(x) + \gamma\phi(x)^2 \tag{48}$$

where α, β, γ related by $\alpha\gamma - \beta^2 = -\frac{1}{4}$ are expressible in terms of the constants a, b, c, d . Let us remark that the equation (41) and the nonlinear superposition principle for its solution (48) coincide with the results recently obtained by W.Schief [12].

Finally, we may show that in the continuum limit $h \to 0$, the equations (47) and (41) lead respectively to the second order linear equation (5) and the nonlinear equation (2). Making in (47) the identification

$$A(x) = -2 + h^2 f(x) \tag{49}$$

we recover in the limit $h \to 0$, the linear differential equation (5). Moreover, setting in (35) $\alpha_1(n) = 0$ and expanding the coefficients of the three last recurrence relations to terms $O(h)$ we obtain in the variable $y(n) \equiv \xi_4(n)$

$$\mathcal{D}^3 y + 4q\mathcal{D}y + 2\frac{\overline{q} - q}{h}\overline{y} = 0, \text{ with } \mathcal{D}y = \frac{\overline{y} - y}{h} \tag{50}$$

whose the continuum limit yields the third order linear equation (4).

On the other hand, expanding $\overline{\overline{y}}$ and $\overline{y}$ up to terms $O(h^2)$ we obtain

$$yy_{xx} - \frac{1}{2}y_x^2 - 2f(x)y^2 = \frac{1}{2}\varepsilon^2. \tag{51}$$

Thus, the difference equation (41) deserves the name of " discrete Ermakov equation " for three reasons:

(i) it is linearisable into a third order linear equation equivalent to a subset of (35) which consist in three first order linear recurrence relations when $\alpha_1(n) = 0$,

(ii) it tends, in the limit $h \to 0$, to the nonlinear differential equation (2),

(iii) its general solution is related to the second order linear difference equation (47) by the quadratic form (48) as in the continuous case.

However, as conjectured in [5], there should exist a discretisation of Pinney equation of degree one, not two like (41). We will now give such an equation.

When one discretises the nonlinear equation (2) following the rules given in [5] one finds a candidate for discrete Ermakov equation linearisable into a third order linear difference equation

$$E \equiv y\mathcal{D}^2\underline{y} - \frac{1}{2}\mathcal{D}y\mathcal{D}\underline{y} + \left(\frac{\lambda_1}{2}(\overline{f} + \underline{f}) + \lambda_2 f\right)\left(\overline{y}\underline{y} + \overline{y}y + \underline{y}y + y^2\right) - c = 0 \quad (52)$$

with $\lambda_1 + \lambda_2 = 1/2$. Indeed, the difference $\mathcal{D}E = (\overline{E} - E)/h$ is given by

$$\mathcal{D}E = \frac{\overline{y} + y}{2}\left[\mathcal{D}^3\underline{y} + \frac{\lambda_1}{2}\left((\overline{\overline{y}} + \overline{y})(\overline{\overline{f}} + f) - (y + \underline{y})(\overline{f} + \underline{f})\right)h^{-1} + \lambda_2\left((\overline{\overline{y}} + \overline{y})\overline{f} - (y + \underline{y})f\right)h^{-1}\right] = 0. \quad (53)$$

We will show here that we obtain the same result, starting from the form (11) in terms of the Schwarzian derivative of X. Although there are a number of discrete forms for this derivative with the property of invariance under the homographic transformation

$$X(x) \to \frac{aX(x) + b}{cX(x) + d}, \quad (54)$$

we will choose a particular form where the discrete analogue of the left hand side of (11) may be transformed into a discrete Riccati equation which is linearisable.

The Schwarzian derivative of the function $X(x)$ of the discrete variable $x = nh$ is defined by:

$$S(x) \equiv 4\frac{[X(x) - X(x-h)][X(x+h) - 3X(x) + 3X(x-h) - X(x-2h)]}{h^2[X(x+h) - X(x-h)][X(x) - X(x-2h)]} - \frac{3}{2}\frac{[X(x+h) - 2X(x) + X(x-h)][X(x) - 2X(x-h) + X(x-2h)]}{h^2[X(x+h) - X(x-h)][X(x) - X(x-2h)]} \quad (55)$$

and corresponds to that given previously by Faddeev and Takhtajan [7]. It is easily seen that this has the correct continuum limit and very interestingly is related to the cross-ratio of four adjacent values of X, i.e.

$$S(x) = -\frac{2}{h^2}\left\{1 - 4\frac{[X(x-h) - X(x-2h)][X(x+h) - X(x)]}{[X(x+h) - X(x-h)][X(x) - X(x-2h)]}\right\} \quad (56)$$