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$$\int_{-\infty}^{\infty} \frac{\log M(t)}{1+t^2}\, dt$$

THE LOGARITHMIC INTEGRAL I

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Notice

In this paperback edition of volume I a number of small errors – and some actual mathematical mistakes – present in the original hard-cover version have been corrected. Many were pointed out to me by Henrik Pedersen, my former student; it was he who observed in particular that the hint given for Problem 28 (b) was ineffective. I wish to express here my gratitude for the considerable service he has thus rendered.

Let me also call the reader's attention to two annoying oversights in volume II. In the statement of the important theorem on p. 65, the condition that the quantities a_k all be > 0 was inadvertently omitted. On p. 406 it would be better, in the last displayed formula, to replace the difference quotient now standing on the right by $\frac{\mu(x+\Delta x)-\mu(x-\Delta x)}{2\Delta x}$.

March 22, 1997
Outremont, Québec

Contents

Preface

The two volumes that follow make up what is meant primarily as a book for reading. One reason for writing them was to give a connected account of some of the ideas that have dominated my mathematical activity for many years. Another, which was to try to help beginning mathematicians interested in analysis learn how to work by showing how I work, seems now less important because my way is far from being the only one. I do hope, at any rate, to encourage younger analysts by the present book in their efforts to become and remain active.

I have loved $\int_{-\infty}^{\infty}(\log M(t)/(1+t^2))\,dt$ – the logarithmic integral—ever since I first read Szegő's discussion about the geometric mean of a function and the theorem named after him in his book on orthogonal polynomials, over 30 years ago. Far from being an isolated artifact, this object plays an important role in many diverse and seemingly unrelated investigations about functions of one real or complex variable, and a serious account of its appearances would involve a good deal of the analysis done since 1900. That will be plain to the reader of this book, where some of that subject's developments in which the integral figures are taken up.

No attempt is made here to treat anything like the full range of topics to which the logarithmic integral is relevant. The most serious omission is that of parts of probability theory, especially of what is called prediction theory. For these, an additional volume would have been needed, and we already have the book of Dym and McKean. Considerations involving H_p spaces have also been avoided as much as possible, and the related material from operator theory left untouched. Quite a few books about those matters are now in circulation.

Of this book, begun in 1983, all but Chapter X and part of Chapter IX was written while I was at McGill University; the remainder was done at UCLA. The first 6 chapters are based on a course (and seminar) given at McGill during the academic year 1982–83, and I am grateful to the mathematics department there for the support provided to me since then out of its rather modest resources. Chapters I–VI and most of the seventh were typed at that department's office.

Chapter VII and parts of Chapter VIII are developed from lectures I gave at the Mittag–Leffler Institute (Sweden) during part of the spring semesters of 1977 and 1983. I am fortunate in having been able to spend almost two years all told working there.

Partial support from the U.S. National Science Foundation was also given me during the first year or two of writing.

I thank first of all John Garnett for having over a long period of time encouraged me to write this book. Lennart Carleson encouraged and helped me with research that led eventually to some of the expositions set out below. I thank him for that and also for my two invitations to the Mittag–Leffler Institute. For the second of those I must also thank Peter Jones who, besides, helped me with at least one item in Chapter VII. The book's very title is from a letter to me by V.P. Havin, and I hope he does not mind my using it. I was unable to think of anything except the mathematical expression it represents!

It was mainly John Taylor who arranged for me to come to McGill in the fall of 1982 and give the course mentioned above. Since then, a good part of my salary at McGill has been paid out of research grants held by him, Jal Choksi, Sam Drury, or Carl Herz. Taylor also came to some of the lectures of my course as did Georg Schmidt. Robert Vermes attended all of them and frequently talked about their material with me. Dr Raymond Couture came part of the time. The students were Janet Henderson, Christian Houdré and Tuan Vu. These people all contributed to the course and helped me to feel that I was doing something of value by giving it. Vermes' constant presence and evident interest in the subject were especially heartening.

Most of the typing for volume I was done by Patricia Ferguson who typed Chapters I through VI and the major part of Chapter VII, and by Babette Dalton who did a very fine job with Chapter VIII. I am beholden to S. Gardiner and P. Jackson of the Press' staff and finally to Dr Tranah, the mathematics editor, for their patience and attention to my desires regarding graphic presentation. The beautiful typesetting was done in India.

August 13, 1987
Laurel, Comté Argenteuil, Québec.

Introduction

The present book has been written so as to necessitate as little consultation by the reader as reasonably possible of other published material. I have hoped to thereby make it accessible to people far from large research centres or any 'good library', and to those who have only their summer vacations to work on mathematics. It is for the same reason that references, where unavoidable, have been made to books rather than periodicals whenever that could be done.

In general, I consider the developments leading up to the various results in the book to be more important than the latter taken by themselves; that is why those developments are set out in more detail than is now customary. My aim has been to enable one to follow them by mostly just reading the text, without having to work on the side to fill in gaps. The reader's active participation is nevertheless solicited, and problems have been given. These are usually accompanied by hints (sometimes copious), so that one may be encouraged to work them out fully rather than feeling stymied by them. It is assumed that the reader's background includes, beyond ordinary undergraduate mathematics, the material which, in North America, is called graduate real and complex variable theory (with a bit of functional analysis). Practically everything needed of this is contained in Rudin's well-known manual. My own preference runs towards a more leisurely approach based on Titchmarsh's *Theory of Functions* and the beautiful *Leçons d'analyse fonctionnelle* of Riesz and Nagy (now available in English). Alongside these books, the use of some supplementary descriptive material on conformal mapping (from Nehari, for instance) is advisable, as is indeed the case with Rudin as well. The Krein–Milman theorem referred to in Chapters VI and X is now included in many books; in Naimark's, for example (on normed algebras or rings), and in Yosida's. In the very few places where more specialized material is called for, additional references will be given. (Exact descriptions of the works just mentioned together with those cited later on can be found in the bibliographies placed at the end of each volume.)

Although the different parts of this book are closely interrelated, they may to a large extent be read independently. Material from Chapter III is, however, called for repeatedly in the succeeding chapters. For finding one's way, the descriptions in the table of contents and the page headings should be helpful; indices to each volume are also provided. Throughout volume I, various arguments commonly looked on as elementary or well-known, but which I nonetheless thought it better to include, have been set in smaller type, and certain readers will miss nothing by passing over them.

The book's units of subdivision are, successively, the chapter, the § (plural §§) and the article. These are indicated respectively by roman numerals, capital letters and arabic numerals. A typical reference would be to '§B.2 of Chapter VI', or to 'Chapter VI, §B.2'. When referring to another article within the same §, that article's number alone is given (e.g., 'see article 3'), and, when it's to another § in the same chapter, just that §'s designation (e.g., 'the discussion in § B') or again, if a particular article in that § is meant, an indication like '§ B.2'. Theorems, definitions and so forth are not numbered, nor are formulas. But certain displayed formulas in a connected development may be labeled by signs like (∗), (†), &c, which are then used to refer to them within that development. The same signs are used over again in different arguments (to designate different formulas), and their order is not fixed. A pause in a discussion is signified by a horizontal space in the text.

About mistakes. There must inevitably be some, although I have tried as hard as I could to eliminate errors in the mathematics as well as misprints. Certain symbols (bars over letters, especially) have an unpleasant tendency to fall off between the typesetters' shop and the camera. I think (and hope) that all the mathematical arguments are clear and correct, at least in their grand lines, and have done my best to make sure of that by rereading everything several times. The reader who, in following a given development, should come upon a misprint or incorrect relation, will thus probably see what should stand in its place and be able to continue unhindered. If something really seems peculiar or devoid of sense, one should try suspending judgement and read ahead for a page or so – what at first appears bizarre may in fact be quite sound and become clear in a moment. Unexpected turnings are encountered as one becomes acquainted with this book's material.

It is beautiful material. May the reader learn to love it as I do.