I

Jensen's formula

On making the substitution $t = \tan(\vartheta/2)$ and then putting $M(t) = P(\vartheta)$, the expression

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log M(t)}{1+t^2}\,dt$$

goes over into

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log P(\vartheta)\,d\vartheta.$$

We begin this book with a discussion of the second integral.

Suppose that $R > 1$ and we are given a function $F(z)$, analytic in $\{|z| < R\}$. If $F(z)$ *has no zeros* for $|z| \leqslant 1$ we can define a *single valued* function $\log F(z)$, analytic for $|z| \leqslant R'$, say, where $1 < R' < R$. By Cauchy's formula we will then have

$$\log F(0) = \frac{1}{2\pi}\int_{0}^{2\pi} \log F(e^{i\vartheta})\,d\vartheta,$$

so, taking the real parts of both sides, we get

$$\log|F(0)| = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|F(e^{i\vartheta})|\,d\vartheta.$$

What if $F(z)$ *has* zeros in $|z| \leqslant 1$? Assume to begin with that there are none *on* $|z| = 1$, and denote those that $F(z)$ does have inside the unit disk by $a_1, a_2, \ldots, a_n$. According to custom, a zero is *repeated according to its multiplicity* in such an enumeration. Put

$$\Phi(z) = \frac{F(z)}{(z-a_1)(z-a_2)\ldots(z-a_n)}.$$

Then $\Phi(z)$ has no zeros in $\{|z| \leqslant 1\}$, so, by the special case already treated,

$$\log|\Phi(0)| = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|\Phi(e^{i\vartheta})|\,d\vartheta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|F(e^{i\vartheta})|\,d\vartheta - \sum_{k=1}^{n} \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|e^{i\vartheta} - a_k|\,d\vartheta.$$

Here we make a side calculation. For $|a_k| < 1$ we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|e^{i\vartheta} - a_k|\,d\vartheta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|1 - \bar{a}_k e^{i\vartheta}|\,d\vartheta,$$

and this $= \log 1 = 0$ by the case already discussed ($F(z)$ without zeros in $|z| \leqslant 1$)! Combined with the previous relation this yields

$$\log|\Phi(0)| = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|F(e^{i\vartheta})|\,d\vartheta.$$

Especially, if $F(0) \neq 0$,

$$\log|F(0)| - \sum_{k=1}^{n} \log|a_k| = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|F(e^{i\vartheta})|\,d\vartheta.$$

The sum on the left can be written differently. *Call $n(r)$ the number of zeros of $F(z)$ in $|z| \leqslant r$ (counting multiplicities).* Then, if $F(0) \neq 0$,

$$-\sum_{k=1}^{n} \log|a_k| = \int_0^1 \frac{n(r)}{r}\,dr.$$

Indeed, since $n(r) = 0$ for $r > 0$ *close* to 0,

$$\int_0^1 \frac{n(r)}{r}\,dr = n(1)\log 1 - \int_0^1 \log r\,dn(r) = -\sum_{k=1}^{n} \log|a_k|.$$

We therefore have

$$\log|F(0)| + \int_0^1 \frac{n(r)}{r}\,dr = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|F(e^{i\vartheta})|\,d\vartheta.$$

In case $F(z)$ is regular in a disk including $\{|z| \leqslant R\}$ in its interior and $F(0) \neq 0$ we can (provided that $F(z) \neq 0$ for $|z| = R$) make a change of variable in the preceding relation and get

$$\log|F(0)| + \int_0^R \frac{n(r)}{r}\,dr = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|F(Re^{i\vartheta})|\,d\vartheta.$$

This is Jensen's formula.

The validity of Jensen's formula *subsists* even when $F(z)$ *has* zeros on the circle $|z| = R$. To see this, observe that then $F(z)$ *will not* have any zeros on the circles $|z| = R'$ with $R' < R$ and sufficiently close to R, for $F(z)$ is analytic in a disk $\{|z| < R + \eta\}$, $\eta > 0$, and not identically zero ($F(0) \neq 0$). So, for such R',

$$\log|F(0)| + \int_0^{R'} \frac{n(r)}{r}\,\mathrm{d}r = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|F(R'\mathrm{e}^{\mathrm{i}\vartheta})|\,\mathrm{d}\vartheta.$$

As $R' \to R$, the left side clearly tends to $\log|F(0)| + \int_0^R (n(r)/r)\,\mathrm{d}r$ – the integral on the left is a *continuous* function of its upper limit because $n(r)$ is *bounded*. We need therefore merely verify that

$$\int_{-\pi}^{\pi} \log|F(R'\mathrm{e}^{\mathrm{i}\vartheta})|\,\mathrm{d}\vartheta \longrightarrow \int_{-\pi}^{\pi} \log|F(Re^{\mathrm{i}\vartheta})|\,\mathrm{d}\vartheta$$

as $R' \to R$. The idea here is the same whether $F(z)$ has *several zeros* on $|z| = R$ or *only one*, and in order to simplify the writing we just treat the latter case. Suppose then that $F(\alpha) = 0$ where $|\alpha| = R$, and there are *no other zeros* in a ring of the form $\{R - \eta \leqslant |z| \leqslant R + \eta\}$, $\eta > 0$. On this ring we then have $|F(z)| \geqslant \text{const.}|z - \alpha|^m$, if m is the multiplicity of the zero at α, so, since $|F(z)|$ is also bounded above there,

$$|\log|F(R'\mathrm{e}^{\mathrm{i}\vartheta})|| \leqslant \text{const.} + m\log^+ \frac{1}{|R'\mathrm{e}^{\mathrm{i}\vartheta} - \alpha|}$$

for $R - \eta \leqslant R' \leqslant R$. (*Here, for* $p > 0$, $\log^+ p$ *denotes* $\log p$ if $p \geqslant 1$ *and* 0 *if* $p < 1$.) The expression on the right is, however, $\leqslant \text{const.} + m\log^+(1/|Re^{\mathrm{i}\vartheta} - \alpha|)$, independently of R', when the latter quantity is close to R:

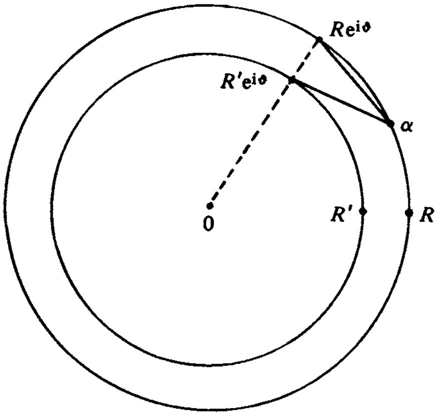

Figure 1

(The constants of course will be *different*; the relation between them need not concern us here.) In other words, for $R' \to R$ the expressions $|\log|F(R'e^{i\vartheta})||$ are bounded above by the *fixed* function const. + $m\log^+(1/|Re^{i\vartheta} - \alpha|)$ of ϑ, *which however, has a finite integral over* $[-\pi, \pi]$, *as we easily check directly*. Since also $\log|F(R'e^{i\vartheta})| \to \log|F(Re^{i\vartheta})|$ pointwise as $R' \to R$, we have

$$\int_{-\pi}^{\pi} \log|F(R'e^{i\vartheta})|\,d\vartheta \longrightarrow \int_{-\pi}^{\pi} \log|F(Re^{i\vartheta})|\,d\vartheta$$

by *Lebesgue's dominated convergence theorem*. This is what we needed to complete our derivation of Jensen's formula. (We see that the *same computation* which shows that

$$\int_{-\pi}^{\pi} \log|F(Re^{i\vartheta})|\,d\vartheta \;>\; -\infty$$

also establishes the *convergence* of $\int_{-\pi}^{\pi}\log|F(R'e^{i\vartheta})|\,d\vartheta$ to that quantity as $R' \to R$!)

Here is a first application of Jensen's formula.

Theorem. *Suppose that* $F(z)$ *is analytic and* $\not\equiv 0$ *for* $|z| < 1$, *and that the integrals*

$$\int_{-\pi}^{\pi} \log^+|F(re^{i\vartheta})|\,d\vartheta$$

are bounded for $0 \leqslant r < 1$. *Then for any* r_0, $0 < r_0 < 1$, *the integrals*

$$\int_{-\pi}^{\pi} \log^-|F(re^{i\vartheta})|\,d\vartheta$$

are bounded for $r_0 < r < 1$.

Notation. For $p \geqslant 0$, we write (as remarked above) $\log^+ p = \max(\log p, 0)$. *We also take* $\log^- p = -\min(\log p, 0)$, *so that* $\log^- p \geqslant 0$ *and* $\log p = \log^+ p - \log^- p$. (Everybody means the same thing by $\log^+ p$, but, regarding $\log^- p$, usage is not uniform.)

Proof of theorem. Without loss of generality (*henceforth abbreviated* 'wlog'), let $F(0) \neq 0$. (Otherwise work with $F(z)/z^k$ for a suitable k instead of $F(z)$.) By Jensen's formula,

$$-\infty < \log|F(0)| \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+|F(re^{i\vartheta})|\,d\vartheta - \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^-|F(re^{i\vartheta})|\,d\vartheta, \quad 0 < r < 1.$$

By hypothesis, the right-hand side is

$$\leqslant \text{ const.} - \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^{-}|F(re^{i\vartheta})|\,d\vartheta.$$

The desired result follows by transposition.

Corollary. *Under the hypothesis of the theorem, suppose that*

$$F(e^{i\vartheta}) = \lim_{r\to 1} F(re^{i\vartheta})$$

exists a.e. Then

$$\int_{-\pi}^{\pi} \log^{-}|F(e^{i\vartheta})|\,d\vartheta \;<\; \infty.$$

Proof. Fatou's lemma.

Remark 1. Actually, the hypothesis of the theorem *forces* a.e. existence of

$$\lim_{r\to 1} F(re^{i\vartheta}).$$

This is a fairly deep result, and depends on Lebesgue's theorem on a.e. existence of derivatives of functions of bounded variation. In the situations we will mostly consider, the existence of this limit can be directly verified ('by inspection'), so the deeper result will not be needed. Therefore we do not prove it now. The interested reader can work up a proof by using the subharmonicity of $\log^{+}|F(z)|$ together with an argument from Chapter III, §F.1, so as to produce a positive measure ν on $[-\pi,\pi]$ for which

$$|F(z)| \;\leqslant\; \left|\exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\vartheta}+z}{e^{i\vartheta}-z}\,d\nu(\vartheta)\right\}\right|, \quad |z|<1.$$

After this, one applies results from §F.2 of Chapter III to the analytic function $\Phi(z)$ within | | on the right, and then to the ratio $F(z)/\Phi(z)$.

Remark 2. The idea of the corollary is that if $|F(z)|$ is *not too big* in $\{|z|<1\}$ (especially if $|F(z)|$ is *bounded* there), then the boundary values $|F(e^{i\vartheta})|$ *cannot be too small* unless $F \equiv 0$.

Problem 1

(a) Let $F(z)$ be entire, $F(0) = 1$, and $|F(z)| \leqslant Ke^{A|z|}$ for all z, where A and K are constants. If $n(R)$ denotes the number of zeros of F having modulus $\leqslant R$, show that, for all R, $n(R) \leqslant eAR + \text{const}$. (Here, the constant depends on K.)

*(b) Show that in the relation established in (a) the coefficient eA of R cannot in general be diminished. (Hint. Fix $R = m/e$ with m a large integer. Compute the maximum value of $(x/R)^{eR}e^{-x}$ for $x \geqslant 0$. Then look at a function which has m equally spaced zeros on the circle $|z| = R$ and no others.)

II

Szegő's theorem

A. The theorem

Szegő's theorem is a beautiful result in approximation theory, obtained with the help of Jensen's formula. Its proof also uses a limit property of integrals involving the Poisson kernel (for the unit disk) which is now taught in many courses on real variable theory. The reader who does not remember that result will find it in §B, together with its proof.

Theorem (Szegő). *Let $w(\vartheta) \geqslant 0$ belong to $L_1(-\pi, \pi)$. Then the infimum of*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|1 - \sum_{n>0} a_n e^{in\vartheta}\right| w(\vartheta)\,d\vartheta,$$

taken with respect to all possible finite sums $\sum_{n>0} a_n e^{in\vartheta}$, is equal to

$$\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\log w(\vartheta)\,d\vartheta\right).$$

Note: $\int_{-\pi}^{\pi}\log^+ w(\vartheta)\,d\vartheta$ *is finite* if $w \in L_1(-\pi, \pi)$. So $\int_{-\pi}^{\pi}\log w(\vartheta)\,d\vartheta$ either *converges*, or else *diverges* to $-\infty$.

Proof of theorem. By the inequality between arithmetic and geometric means,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|1 - \sum_{n>0} a_n e^{in\vartheta}\right| w(\vartheta)\,d\vartheta$$
$$\geqslant \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\log\left|1 - \sum_{n>0} a_n e^{in\vartheta}\right| + \log w(\vartheta)\right)d\vartheta\right\}.$$

Jensen's formula applied to $F(z) = 1 - \sum_{n>0} a_n z^n$ shows that this last expression is always

$$\geqslant \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\log w(\vartheta)\,d\vartheta\right);$$

the desired infimum is thus $\geqslant$ the latter quantity. We must establish the reverse inequality.

Write $w_N(\vartheta) = \max(w(\vartheta), e^{-N})$. By Lebesgue's monotone convergence theorem and the finiteness of $\int_{-\pi}^{\pi} \log^+ w(\vartheta)\,d\vartheta$ we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log w_N(\vartheta)\,d\vartheta \underset{N}{\longrightarrow} \frac{1}{2\pi}\int_{-\pi}^{\pi} \log w(\vartheta)\,d\vartheta.$$

It will therefore be enough to show that for any N and any $\delta > 0$ there exists some finite sum $1 - \sum_{k>0} A_k e^{ik\vartheta}$ such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|1 - \sum_{k>0} A_k e^{ik\vartheta}\right| w(\vartheta)\,d\vartheta \;<\; \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log w_N(\vartheta)\,d\vartheta\right) + \delta.$$

To this end, put first of all

$$(*)\qquad F_N(z) = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\log\left(\frac{1}{w_N(t)}\right)dt\right\}$$

for $|z| < 1$. We have

$$(\ast\ast)\qquad F_N(0) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left(\frac{1}{w_N(t)}\right)dt\right).$$

Since $w_N(t) \geqslant e^{-N}$, $|F_N(z)| \leqslant e^N$ for $|z| < 1$. Indeed, taking real parts of the logarithms of both sides of $(*)$ gives us

$$\log|F_N(re^{i\vartheta})| = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1-r^2-2r\cos(\vartheta-t)}\log\left(\frac{1}{w_N(t)}\right)dt.$$

On the right side we recognize the *Poisson kernel* (that's the *real reason* for using $(e^{it}+z)/(e^{it}-z)$ in $(*)$, aside from the fact that we *want* $F_N(z)$ to be *analytic* in $\{|z|<1\}$). As one knows,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(\vartheta-t)}\,dt = 1;$$

the integrand is obviously *positive*. We see that $\log|F_N(re^{i\vartheta})| \leqslant N$ by the previous formula.

Now we use another, much *finer* property of the Poisson kernel, established in §B below. According to the latter,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(\vartheta-t)}\log\left(\frac{1}{w_N(t)}\right)dt \longrightarrow \log\left(\frac{1}{w_N(\vartheta)}\right)$$

for almost all ϑ as $r \to 1$. So $|F_N(re^{i\vartheta})| \to 1/w_N(\vartheta)$ a.e. for $r \to 1$. However, $|F_N(z)|$ is bounded above and $w(\vartheta) \in L_1(-\pi,\pi)$. Therefore, by *dominated* convergence,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|F_N(re^{i\vartheta})|w(\vartheta)\,d\vartheta \longrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{w(\vartheta)}{w_N(\vartheta)}\,d\vartheta$$

as $r\to 1$. The right-hand side is *clearly* $\leqslant 1$. Given $\varepsilon > 0$ we can therefore get an $r<1$ with

$$(\dagger)\qquad \frac{1}{2\pi}\int_{-\pi}^{\pi}|F_N(re^{i\vartheta})|w(\vartheta)\,d\vartheta < 1+\varepsilon.$$

Fix such an r.

By the very *form* of the right side of $(*)$, $F_N(z)$ is *analytic* in $\{|z|<1\}$; it therefore has a Taylor expansion there. And, by $(\ast\ast)$, $F_N(0)\neq 0$. Letting $S(z)$ be any *partial sum* of the Taylor series for $F_N(z)$, we see that *for our fixed* r,

$$\frac{S(re^{i\vartheta})}{F_N(0)} \quad \text{is of the form } 1-\sum_{k>0}A_k e^{ik\vartheta},$$

the sum on the right being finite. Since $F_N(z)$ is regular in $\{|z|<1\}$ and $r<1$, we see by $(\dagger)$ that we can choose the partial sum $S(z)$ so that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|S(re^{i\vartheta})|w(\vartheta)\,d\vartheta < 1+2\varepsilon.$$

Hence

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|1-\sum_{k>0}A_k e^{ik\vartheta}\right|w(\vartheta)\,d\vartheta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{S(re^{i\vartheta})}{F_N(0)}\right|w(\vartheta)\,d\vartheta \;\leqslant\; (1+2\varepsilon)\cdot\frac{1}{F_N(0)},$$

which equals

$$(1+2\varepsilon)\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\log w_N(t)\,dt\right)\quad \text{by } (\ast\ast).$$

This is enough, and we are done.

Remark. This most elegant result was extended by Kolmogorov, and then by Krein, who evaluated the infimum of

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|1-\sum_{n>0}a_n e^{in\vartheta}\right|d\mu(\vartheta)$$

for all finite sums $\sum_{n>0}a_n e^{in\vartheta}$ when μ is *any finite positive measure*. It turns out that the singular part of μ (with respect to Lebesgue measure) *has no influence here*, that the infimum is simply equal to

$$\exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left(\frac{d\mu(\vartheta)}{d\vartheta}\right)d\vartheta\right\}.$$

I do not give the proof of this result. It depends on the construction of Fatou–Riesz functions which, while not very difficult, is not really part of the material being treated here. The interested reader may find a proof in many books; some of the older ones which have it are Hoffman's and Akhiezer's (on approximation theory). The newer books by Garnett (on bounded analytic functions), and by me (on H_p spaces) both contain proofs.

B. The pointwise approximate identity property of the Poisson kernel

Theorem. Let $P(\vartheta) \in L_1(-\pi, \pi)$, *and, for* $r < 1$, *write*

$$U(re^{i\vartheta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-r^2}{1+r^2-2r\cos(\vartheta-t)}P(t)\,dt.$$

For almost every ϑ, $U(z)$ *tends to* $P(\vartheta)$ *uniformly as* z *tends to* $e^{i\vartheta}$ *within any sector of the form*

$$|\arg(1-e^{-i\vartheta}z)| \leqslant \alpha < \frac{\pi}{2}.*$$

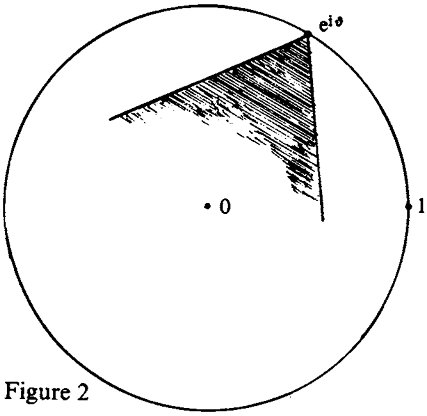

Figure 2

Remark. We write '$U(z) \to P(\vartheta)$ a.e. for $z \xrightarrow{\angle} e^{i\vartheta}$.' Some people say that $U(z) \to P(\vartheta)$ a.e. for z tending *non-tangentially* to $e^{i\vartheta}$, others say that

* It is clear that for z of modulus $> \sin\alpha$ in such a sector we have $|\arg z - \vartheta| \leqslant K(1-|z|)$ with a constant K depending on α.