Introduction to Geometric Probability

This is the first modern introduction to geometric probability, also known as integral geometry. The subject is presented at an elementary level, requiring little more than first year graduate mathematics. The theory of intrinsic volumes due to Hadwiger, McMullen, Santaló and others is presented, along with a complete and elementary proof of Hadwiger's characterization theorem of invariant measures in Euclidean n-space. The theory of the Euler characteristic is developed from an integral-geometric point of view. The authors then prove the fundamental theorem of integral geometry, namely the kinematic formula. Finally the analogies between invariant measures on polyconvex sets and measures on order ideals of finite partially ordered sets are investigated. The relationship between convex geometry and enumerative combinatorics motivates much of the presentation. Every chapter concludes with a list of unsolved problems. Geometers and combinatorialists will find this a stimulating and fruitful tale.

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Preface

If we were allowed to rename the field of geometric probability – sometimes already renamed integral geometry – then we would be tempted to choose the oxymoron 'continuous combinatorics.' On more than one occasion the two fields, geometric probability and enumerative combinatorics, are brought together by mathematical analogy, that most effective breaker of barriers.

Like combinatorial enumeration, where sequences of objects bearing a common feature are unified by the idea of a generating function, geometric probability studies sets of geometric objects bearing a common feature, which are unified by the idea of an invariant measure. The basic idea is extremely simple. When considering straight lines, pairs of points, or triangles in space, one determines the invariant measure on the variety of straight lines, of pairs of points, of triangles. This idea is strangely reminiscent of the underlying idea of enumerative geometry, with one major difference: whereas enumerative geometry is bound to the counting of finite sets, geometric probability is given greater freedom, by extending the concept of enumeration to allow the assigning of invariant measures. Invariant measures are far easier to compute and, we dare add, more useful than the curiously large integers that are computed in enumerative geometry. This basic idea goes back to Crofton's article in the ninth edition of the Encyclopaedia Britannica, an article that created the subject from scratch and that is still worth reading today. The one other brilliant contribution to geometric probability in the past century was Barbier's solution of the Buffon needle problem, which remains to this day the basic trick of the subject, still being secretly exploited in ever unsuspected ways.

Geometric probability has suffered in this century the fate of other fields that would have enjoyed a healthy autonomous development, had it not been for the overpowering development of representation theory. One can reduce integral geometry to the study of actions of Lie groups, to symmetric spaces, to the Radon transform; in so doing, however, the authentic problematic of the subject is lost. Geometric probability is a customer of representation theory, in the same sense that mechanics is a customer of the calculus.

The purpose of this book is to present the three basic ideas of geometric probability, stripped of all reliance on group-theoretic techniques. First, we investigate measures on polyconvex sets (i.e., finite unions of compact convex sets) in Euclidean spaces of arbitrary dimension that are invariant under the group of Euclidean motions. A great many mathematicians are still basking in the illusion that there is only one such measure, namely, the volume. We merrily destroy this illusion by proving what is at present the fundamental result of the field (due to Hadwiger), stating that the space of such invariant measures is of dimension $n+1$ in a Euclidean space of dimension n. The proof of this fundamental result given in the text is new, due to the first author. It becomes clear, on reading the applications of the fundamental theorem, that the basic invariant measure to be singled out from such a bounty is not the volume, but the Euler characteristic (as Steve Schanuel was first to realize). Here again we meet with wide ignorance on the part of the mathematical public: the fundamental fact that the Euler characteristic is an invariant measure (in fact, it is the only integer-valued invariant measure) is not as well known as it should be. It leads to one-line proofs of most of the fundamental theorems on convex sets. We develop the theory of the Euler characteristic from scratch, in a way that makes it look like an ordinary integral.

Second, we prove the fundamental formula of integral geometry, viz., the kinematic formula. Here we displace the common device of Minkowski sums from its typically central role, not merely as a display of mathematical machismo, but with an ulterior motive.

Third, we try to bring out from the beginning the striking analogy between the computation of invariant measures and certain combinatorial properties of finite partially ordered sets. The second author pointed out in 1967 that the notion of Euler characteristic could be extended to such partially ordered sets by means of the Möbius function. We now go one step further and show that an analogue of the theory of invariant measures in Euclidean space can be worked out in partially ordered sets, including finite analogues of the kinematic formula and even of Helly's theorem. This analogy brings out in stark contrast the unexplored terrain of classical geometric probability, namely, a thorough understanding of the integral geometric structure of the lattice of subspaces of Euclidean space under the action of the orthogonal group. It also brings us closer to the current outer limits of mathematics, to the theory of Hecke algebras, to Schubert varieties and to the quantum world.

We hope that the reading of this introduction to the field of geometric probability will encourage further development of these analogies.

The text is based on the 'Lezioni Lincee' given by the second author in 1986 at the Scuola Normale Superiore in Pisa. The authors wish to thank Ennio De Giorgi, Edoardo Vesentini, and Luigi Radicati for providing an interested audience for the original lectures. Thanks are also due to Stefano Mortola for his careful reading of the initial draft, and to Beifang Chen, Steve Fisk, Joseph Fu, Steven Holt, Erwin Lutwak, and three anonymous referees for their valuable comments and suggestions.

Using this book

Although parts of this book assume a knowledge of basic point-set topology, measure theory, and elementary probability theory, the greater part of the text should be accessible to advanced undergraduates. Proofs are either given in full or else stretched to the point from which the reader will be able to reconstruct them without effort. Only on certain technical measure-theoretic points have we felt the need to omit details that, although indispensable in a detailed treatment, are of questionable relevance in an exposition that is meant to stress geometric insight and combinatorial analogy. Some notions that appear vague in the early sections will be revisited later on, after language has been developed for a treatment in clear and rigorous terms. References and open problems are deferred to the notes at the end of each chapter.