Abstracts of the talks

Short Courses

Fields of Definition of Covers; Embedding Problems over Large Fields. *Pierre Dèbes.*

I. The first talk deals with joint work with Jean-Claude Douai. Let $f : X \to B$ be a finite cover defined over the separable closure K_s of a field K, with B an algebraic variety defined over K. Assume that f is *isomorphic* to each of its conjugates under $G(K_s/K)$. The field K is called the *field of moduli* of the cover. Does it follow that the given cover can be defined over K? The answer is "No" in general: there is an obstruction to the field of moduli being a field of definition. Still, how can the obstruction be measured? We present a general approach for this problem. The obstruction is entirely of a cohomological nature. This was known only in the case of G-covers, *i.e.*, Galois covers given together with their automorphisms. This special case happens to be the simplest one. In the situation of *mere* covers, the problem is shown to be controlled not by one, as for G-covers, but by several characteristic classes in $H^2(K, Z(G), L)$ (for a certain action L of $G(K_s/K)$ on the center $Z(G)$ the group of the cover). Furthermore our approach reveals a more hidden obstruction coming on top of the main one, called the first obstruction and which does not exist for G-covers.

Our Main Theorem yields quite concrete criteria for the field of moduli to be a field of definition. Such criteria were not available in the general situation of mere covers. Furthermore the base space B can be here an algebraic variety of any dimension and the ground field K a field of any characteristic. All classical results, for which the base space was the projective line $\mathbf{P}^1$ over $\overline{\mathbb{Q}}$, are contained as special cases.

Our Main Theorem also leads to some local-global type results. For example we prove this *local-to-global* principle: a G-cover $f : X \to B$ is defined over $\mathbb{Q}$ if and only if it is defined over $\mathbb{Q}_p$ for all primes p. This was conjectured by E. Dew and proved by the author in the special case of G-covers of $\mathbf{P}^1$. We will develop this local-to-global result and other related questions. We will prove in particular this *global-to-local* principle for covers (or G-covers): if a cover (or a G-cover) $f : X \to B$ has a number field K as field of moduli, then it may not be defined over K, but it is necessarily defined over all but finitely many completions K_v of K.

II. The Inverse Galois Problem — is each finite group a quotient of $G(\mathbb{Q})$? — and Safarevic's conjecture — $G(\mathbb{Q}^{ab})$ is a free profinite group — are two main questions generally asked about the absolute Galois group $G(\mathbb{Q})$ of $\mathbb{Q}$.

A fruitful approach consists in studying the action of $G(\mathbb{Q})$ on covers of $\mathbf{P}^1$ defined over $\overline{\mathbb{Q}}$, or equivalently, on extensions of $\overline{\mathbb{Q}}(T)$.

The goal of this talk is twofold. First we present a "Main Conjecture" that contains all basic conjectures of the area, including the Inverse Galois Problem and Safarevic's conjecture but also the Fried-Völklein conjecture — $G(K)$ projective + K hilbertian $\Rightarrow G(K)$ pro-free — . A special case of the Main Conjecture is that, if K is any given field, then all split embedding problems for $G(K(T))$ have a strong solution. The general form of the Main Conjecture is that the same is true for split embedding problems with "some extra constraint on $\overline{K}$".

In the second part, we present a "Main Theorem" that unifies a whole series of results that have recently appeared about the absolute Galois group $G(K(T))$ when K is a algebraically closed, or Pseudo Algebraically Closed, or is a local field, or is Pseudo S Closed (*e.g.* the field of totally real (or p-adic) algebraic numbers). The Main Theorem is that the Main Conjecture is true if the field K is *large*. By definition, a field K is large if each smooth geometrically irreducible curve defined over K has infinitely many K-rational points provided that there is at least one. The examples above (K algebraically closed, etc.) are examples of large fields.

The main contributions to the Main Theorem are due to D. Harbater who introduced some very efficient "patching and glueing techniques" for covers of $\mathbf{P}^1$, F. Pop for the arithmetic ingredients of the proof, in particular, his work on the property "K large" and M. Fried, for the idea of working on *families* of covers. Q. Liu, H. Völklein, D. Haran and the author were other contributors to this result.

Coordinates on Teichmüller and moduli space. *Adrien Douady.*

I. In the first lecture, pants decompositions (also known as maximal multicurves) of a surface are described, together with some elementary topological and enumerative properties. We then show how, picking a fixed decomposition, one can define the corresponding Fenchel-Nielsen coordinates which provide a global *real* analytic coordinatization of Teichmüller space. These coordinates can also be used in the study of the moduli spaces and their compactifications.

II. The second lecture is devoted to an introduction to the *complex* analytic theory. Starting from the definition and existence statement of Strebel quadratic differentials, we show how to introduce deformation parameters which afford a local complex analytic coordinatization of Teimüller space. More specifically, to a quadratic differential are associated horizontal and vertical foliations; picking one of them (this is a matter of convention), one associates to a Strebel differential a decomposition of the underlying Riemann surface into cylinders. The deformation parameters which are used in the coordinatization may be intuitively described as the mutual sliding and twisting of these cylinders, or more visually "stove pipes". Once these holomorphic coordinates on Teichmüller space are obtained, they descend to the moduli space, thus providing one of the several ways to endow this last space with a complex structure.

Topological field theory and connections with CFT, and the topology of configuration spaces. *Ruth Lawrence.*

I. This short course aims to show some of the structures naturally arising from topological, geometric and combinatorial approaches to topological quantum field theory as well as relations between them. The first lecture concentrates on the definition of a topological field theory and shows how simple topology relating to decompositions of manifolds into 'elementary pieces' forces the existence of a tight algebraic structure for TQFTs in two and three dimensions.

II. In the second lecture, we concentrate on the connections with conformal field theory and the geometry of configuration spaces of points. In particular, we see that the combinatorics involved in the representation theory of quantum groups forms a bridge between the two approaches.

On universal monodromy representations of Galois-Teichmüller modular groups. *Hiroaki Nakamura.*

I. In this course, I explain the basic setting and recent advanced results in the study of pro-ℓ exterior Galois representations arising from algebraic curves. When the algebraic curve varies, the Galois representation is deformed, but it turns out that there is an invariant portion common to all algebraic curves. These towers are defined by the universal monodromy representations of Galois-Teichmüller modular groups together with the weight filtrations in the so-called pro-ℓ mapping class groups. The problem of showing stability properties of these fields of definition with respect to genera and marking points was posed by Takayuki Oda, and recently fairly established by cooperation of Y. Ihara, M. Matsumoto, N. Takao, R. Ueno and the author.

II. In the second part, we explain graphs of profinite groups associated with combinatorial data on maximally degenerate curves.

A survey of Grothendieck-Teichmüller theory. *Leila Schneps.*

I. The first part of this short course surveys recent work on the following topics: new properties of the Grothendieck-Teichmüller group GT analogous to known properties of the absolute Galois group known to be contained in it and conjecturally equal to it, such as torsion, behavior of complex conjugation, action on combinatorial structures such as dessins and braid groups, and others, particularly the Galois-compatible action of GT on various fundamental groupoids of geometric objects, particularly moduli spaces of Riemann surfaces with marked points.

II. In the second part, we consider universal Teichmüller theory. We introduce the isomorphism (proved by M. Imbert) between Richard u Thompson's group, the group $\mathrm{PPSL}_2(\mathbb{Z})$ of piecewise $\mathrm{PSL}_2(\mathbb{Z})$-transformations, and Penner's universal Ptolemy group G. Penner's presentation of this group emphasizes a remarkable pentagon equation which indicates a connection with the Grothendieck-Teichmüller group. This connection is made explicit via the result that $\widehat{GT}$ acts on a suitable *groupoid-profinite* completion of the universal Ptolemy group.

Individual talks

Characterizing curves by their dessins. *Jean-Marc Couveignes.* We recall that a Belyi function ϕ is a function from some curve $\mathcal{C}$ defined over $\overline{\mathbb{Q}}$ unramified outside $\{0, 1, \infty\}$. Then $(\mathcal{C}, \phi)$ is called a Belyi pair. A morphism between two such pairs is a map I from $\mathcal{C}_1$ to $\mathcal{C}_2$ such that $\phi_2 I = \phi_1$. An isomorphism class of Belyi pairs is called a dessin following Grothendieck. We first give a sketch of proof for some slight improvement of Belyi's theorem stating the existence of a Belyi function with no automorphisms on any curve defined over a number field $\mathbf{K}$. To each curve $\mathcal{C}$ defined over $\mathbf{K}$ we associate a dessin $(\mathcal{C}, \phi)$ with no automorphisms (or the set of all of them). This is enough to characterize this curve up to isomorphisms defined over $\mathbf{K}$. We then look for some explicit examples in order to test which kind of arithmetic information on $\mathcal{C}$ can be read on the topological structure of its characteristic dessins. We insist that a naive use of Belyi's theorem is not likely to provide any such non trivial example. Instead, we obtain families of dessins by an indirect way, using coverings ramified over four points and the corresponding moduli spaces (called Hurwitz spaces after Fried). For example, we present a family of dessins of genus zero with no automorphims, indexed by four parameters m, n, p, q, and associated to the curves $\mathcal{C}_{m,n,p,q}$ in $\mathbf{P}_3$ given by the equations $ma + nb + pc + qd = 0$ and $ma^2 + nb^2 + pc^2 + qd^2 = 0$. We give both the algebraic and topological description of these dessins and prove that any conic can be set in the form of some $\mathcal{C}_{m,n,p,q}$ for suitables values of the parameters. We finish by a brief study of the case $m = 1$, $n = 2$, $p = 3$, $q = 7$ where we prove that the conic has bad reduction at 7 while 7 is prime to the degree of and any ramification in our associated dessin.

Presentation of the Mapping Class Group. *Sylvain Gervais.* The Mapping Class Group $M(g, n)$ of a genus g surface S with n boundary components is the group of diffeomorphisms of S which leave fixed its boundary, modulo whose which are isotopic to the identity. $M(g, n)$ is generated by Dehn twists for all g and n; we will give a presentation of $M(g, n)$ considering all twists as generators. When g is greater than or equal to 2, $M(g, n)$ is generated by twists along non-separating curves. A presentation is also given with these generators. In both cases, all the relations live in surfaces of genus 0 with 4 boundary components or of genus 1 with 1 or 2 boundary components.

Arithmetic aspects of Teichmüller theory. *Bill Harvey.* We focus attention on two ways in which the theory of uniformisation for Riemann surfaces X which are hyperbolic (covered by the real hyperbolic disc D^2) has points of contact with fields of definition for X as an algebraic curve.

1) Following Belyi's theorem, we know that X can be defined over $\overline{\mathbb{Q}}$ if and only if a Fuchsian uniformising group for X is contained in a triangle group. For many (perhaps all) such surfaces, there exist holomorphic quadratic differential forms ω on X such that the associated Teichmüller deformation disc $\mathbf{D}([X], \omega) \subseteqq T(X)$, (here $T(X)$ denotes the Teichmüller space of the closed surface X), has the property that the subgroup of the mapping class group $\mathrm{Mod}(X)$ which preserves $\mathbf{D}$ is a Fuchsian group (of the first kind) intermediate between a triangle group of type $\{p, q, \infty\}$ and a subgroup representing $\pi_1(X^*)$, the surface X with the zero set of ω removed.

2) The classical Schottky uniformisation of a (complex) Riemann surface has an analogue for non-Archimedean fields due to Mumford. This motivates the search for a choice of Schottky uniformisation of curves definable over a given number field. If we choose a class of stable degeneration for a genus g curve (via its dual graph), the work of Bers and Maskit provides complex coordinates for a natural uniformisation of such curves. I have shown

Theorem. A modification of Maskit's coordinates determines Schottky coordinates which define Mumford curves at a specific set of places of the field generated by the matrix entries of the (classical) Schottky group.

A complex interpretation of Fricke coordinates. *John Hubbard.* No abstract submitted.

On π_1 of curves over arithmetic ground fields. *Yasutaka Ihara.* Let X be a smooth irreducible curve over a field k (assumed to be finite over a prime field). The first topic is a simple but useful observation of A. Tamagawa. When k is finite and $g(X) > 0$, he characterized group-theoretically which section $s : \mathrm{Gal}(\bar{k}/k) \to \pi_1(X)$ of the projection $\pi_1(X) \to \mathrm{Gal}(\bar{k}/k) \simeq \hat{\mathbb{Z}}$ comes from a k-rational point $P \in X(k)$. The second subject is the quotient of $\pi_1(X)$ by the normal subgroup generated by all conjugates of $s(\mathrm{Gal}(\bar{k}/k))$, where s is a section corresponding to some P. This quotient, denoted by $\pi_1(X)_{(P)}$, is the Galois group of the tower of finite etale covers of X in which P splits completely. We first discuss what can be said in general about this group $\pi_1(X)_{(P)}$ (various "derived" quotient groups must be "small", etc.) using "abelian" mathematics, and then restrict to Shimura curves to discuss deeper phenomena.

A cohomological interpretation of the Grothendieck-Teichmüller group. *Pierre Lochak.* The result given in this talk, following from joint work with L. Schneps, is the following. The three defining relations of the Grothendieck-Teichmüller group are cocycle relations for certain non-commutative cohomology sets of cyclic groups with values in braid groups. These cohomology sets are very simple, and one can actually compute explicit coboundaries representing the elements of GT. The methods for calculating the cohomology sets are due to Serre, Brown and Scheiderer; the same methods also gives the result that complex conjugation is self-normalizing in GT.

Etale covers of a generic curve in characteristic $p > 0$ and ordinarity. *Michel Matignon.* Contrary to the affine case (Abhyankar's conjecture) there is no conjecture concerning finite quotients of the fundamental group of a smooth projective curve. The fundamental group codes the Hasse-Witt invariants of etale covers; this gives an infinite set of invariants whose study was begun by H. Katsurada and S. Nakajima. In this talk we concentrate on π_1 of a generic curve of genus g over an algebraically closed field k of characteristic $p > 0$; such a curve degenerates into proper stable curves over k of arithmetic genus g; then a theorem due to M. Saïdi gives the profinite fundamental group of the graph of groups, built on the intersection graph of the stable curve and with group data the tame fundamental group of irreducible components minus the intersection points and amalgamation along tame inertia at these points, as a quotient of π_1. The case of maximal degenerations is related to $\pi_1^t(\mathbb{P}^1 - \{0,1,\infty\})$. An optimistic conjecture is that every etale cover of a generic curve is ordinary; this is known for abelian covers. A first attempt towards this conjecture is the study of covers of the Legendre elliptic curve which are étale outside ∞ (such covers appear by degenerating into a chain of generic elliptic curves); one expects that those which don't factorise through an isogeny are still ordinary; the case of hyperelliptic and (in particular) genus two curves is examined. In the last part we look at the geometry of the "Deligne-Mumford" boundary of the Teichmüller tower of M_g; in this way one hopes to give a group theoretic description of quotients of π_1 which give rise to ordinary etale covers.

Galois actions on braid-like groups. *Makoto Matsumoto.* Let V be a geometrically connected variety defined over a number field k. Then, by Grothendieck's theory, we have an outer action of the absolute Galois group G_k of k on the profinite completion of the topological fundamental group of $V(\mathbf{C})$. We denote this representation by ρ_V. By Belyi's theorem, this action is faithful if V is the projective line minus three points, and each element σ of G_k has its unique "coordinate" $(\chi(\sigma), f_\sigma)$ in $\hat{\mathbb{Z}} \times [F_2, F_2]$. The basic question is: can we use this coordinate in order to describe ρ_V for V other than projective line minus three points? This occured in Ihara's geometric proof of the embedding of $G_\mathbb{Q}$ into $\hat{GT}$ (done by Drinfeld's) using two-dimensional tangential base point.

We show two examples where same kinds of arguments can be applied: (1) V configuration space of some points on an open curve, and (2) moduli space of curves of genus g, with $g = 2, 3$. The former example can be applied to show a generalization of Belyi's Injectivity theorem for nonabelian affine curves. The latter example for $g = 3$ uses a close relationship between the deformation space of E_7-singularity and the moduli stack of genus 3 curves.

On Grothendieck's anabelian conjecture. *Florian Pop.* The idea of Grothendieck's anabelian geometry is that under certain "anabelian" hypotheses the isomorphy type, hence the geometry and the arithmetic of schemes, is encoded in their étale fundamental groups. Relatively speaking, if S is some base scheme, and $\mathcal{A}$ is an anabelian category of S-schemes, then for every object X and Y of $\mathcal{A}$ one should have the following

$$\mathrm{Isom}(X,Y) \cong \mathrm{Out}_{\pi_1(S)}(\pi_1(X), \pi_1(Y))$$

in a functorial way. Here π_1 denotes the étale fundamental group, Isom denotes the space of all $\mathcal{A}$-isomorphisms, and Out denotes the space of all outer isomorphisms.

Richard Thompson's chameleon groups: old and new. *Vlad Sergiescu.* Around 1965, Richard J. Thompson (unpublished) constructed two finitely presented infinite simple groups. They can be defined as groups of affine dyadic homeomorphisms (resp. interval exchanges) of S^1. Recently, it became clear that these groups relate to the universal Ptolemy groupoid constructed by R. J. Penner. The talk surveys these and other developments leading to the recent Lochak-Schneps construction of an action of the Grothendieck-Teichmüller group on an enlarged braids version of Ptolemy; suitably completed.

Singularities of moduli spaces of curves, new Galois invariants of Grothendieck dessins and finite projective geometry. *George Shabat.* In this talk, reflecting joint work with N. Adrianov, we discuss the connections between the Grothendieck *dessins d'enfant* theory and the moduli families of curves. An explicit formula for a Strebel differential on any curve corresponding to a given dessin is suggested. The arithmetical nature of curves with many automorphisms is mentioned.

The equivalence between the classical and the *cartographical* Galois theories is established. From this point of view for a particular case of genus 0 dessins with one cell a new invariant, the *edge rotation group*, is introduced. The examples of its non-trivial behaviour, including the occurrence of two Mathieu groups, are presented. The list of examples when this group is finite projective is discussed.

On groups acting on dessin-labeled objects. *Vasily Shabat.* Grothendieck's theory of *dessins d'enfant* offers a possibility to visualize the action of Galois group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on Belyi pairs.

We use a purely combinatorial approach and introduce the *edge group* that acts on dessins with one marked oriented edge. To do this we define an operation *semiflip* and use it together with known operations of cartographic group. The action of the edge group on the set of dessins of fixed genus and fixed number of edges with one marked flag is transitive. We also discuss an alternative definition of semiflip that preserves the number of vertices (and cells). The well-known group generated by elementary moves of ideal triangulations turns out to be a subgroup of the edge group. Since triangulations parametrize cells of cellular decomposition of Teichmuller space and moduli space, we may speculate that the moves between cells can be realized in terms of (semi)flips. We also discuss relations within the edge group.

Non-abelian unipotent periods. *Zdislaw Wojtkowiak.* We study the monodromy of the universal unipotent connection on $V = \mathbf{C} \setminus \{0,1\}$. To the monodromy homomorphism we associate a certain torsor and we calculate this torsor partially. The group corresponding to this torsor is closely related to the image of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ in $\mathrm{Aut}(\pi_1(V)_{\mathrm{et}})$.

Hypergeometric functions and moduli of abelian varieties. *Jürgen Wolfart.*

In this talk it was shown how monodromy groups of classical hypergeometric functions and Appell Lauricella functions, i.e. in particular Fuchsian triangle groups and Picard - Terada - Mostow - Deligne groups can be seen as modular groups of suitable families of complex abelian varieties, even in the case where the groups in question are not arithmetic. One obtains so-called modular embeddings of the groups into modular groups acting on higher dimensional complex symmetric domains compatible with analytic embeddings of the domains themselves. Singular points of the differential equations play a particular role for complex multiplication of the parametrized abelian varieties. (Joint work with Paula Beazley Cohen, Ann. ENS 1993)

Evening seminar on Teichmüller and moduli space.

Decompositions of surfaces and compactification of moduli space. *Xavier Buff*

Cellulation de l'espace des modules. *Jean Nicolas Dénarié.*

Fuchsian Model of Teichmüller Space according to Imayoshi. *Ivan Faucheux.*

Penner's cells are cells. *Jérôme Fehrenbach.*

Part I. Dessins d'enfants

Unicellular cartography and Galois orbits of plane trees

Nicolai Adrianov and George Shabat

Galois groups, monodromy groups and cartographic groups

Gareth Jones and Manfred Streit

Permutation techniques for coset representations of modular subgroups

Tim Hsu

Dessins d'enfants en genre 1

Leonardo Zapponi