Introduction

All the papers presented here come into the category of incidence structures. Overwhelmingly, they have a geometrical point of view, which both motivates and illustrates.

Although all the topics of all the papers are on interconnected themes, they are here subdivided into categories.

A $t-(v,k,\lambda)$ *design* is an incidence structure $\mathcal{S}=(\mathcal{P},\mathcal{B},\mathrm{I})$ such that

(a) $\mathcal{P}$ is a set of v points;

(b) each block B in $\mathcal{B}$ is incident with a fixed number k of points;

(c) any t points are incident with precisely λ blocks.

The *order* of a design is $n=k-\lambda$. A *Steiner system* is a $t-$design with $\lambda=1$, and a *Steiner triple system* also has $t=2, k=3$.

Now let $\mathcal{S}$ be a connected incidence structure of points and lines such that the lines cover the points and such that each line contains at least two points. Suppose also that

(a) there are $s+1$ points on a line, with $s>0$;

(b) there are $t+1$ points through a point, with $t>0$;

(c) two points are incident with at most one line;

(d) two lines are incident with at most one point;

(e) given a non-incident point-line pair, there are α lines through the point meeting the line, with $\alpha>0$.

Then $\mathcal{S}$ is a *partial geometry*; in particular, when $\alpha=1$, it is a *generalized quadrangle*.

Geometries and groups

Cameron contributes a magisterial survey of the current state of finite geometry from a group-theoretical viewpoint; this essay alone is worth the price of the volume!

Designs

A continuing problem is to find $t-$designs for a given t with small k and λ.

Betten, *Laue* and *Wassermann* find new $7-$designs on the projective lines $PG(1,23)$, $PG(1,25)$, $PG(1,32)$ by computation. They have parameters as follows:

(a) $7-(24,8,\lambda)$, for $\lambda = 4,\ 5,\ 6,\ 7,\ 8$;

(b) $7-(24,9,\lambda)$, for $\lambda = 40,\ 48,\ 64$;

(c) $7-(26,8,6)$;

(d) $7-(26,9,\lambda)$, for $\lambda = 54,\ 63,\ 81$;

(e) $7-(33,8,10)$.

Brouwer, *Haemers* and *Tonchev* consider whether, given a partial geometry $\mathcal{P}$ with v points and k points on a line, one can add to the line set a set of $k-$subsets of points such that the extended family of $k-$subsets is a $2-(v,k,1)$ design (or a Steiner system $S(2,k,v)$). Various conditions are given and it is shown in particular that the partial geometry $PQ^+(4m-1,2)$ is embeddable in a Steiner $2-$design if and only if $m \leq 2$.

A biplane is a symmetric $2-$design with $\lambda = 2$, and are rare objects indeed. *Key* and *Tonchev* consider the five known biplanes of order 9, that is $2-(56,11,2)$ designs, and investigate them via the associated ternary codes. The computations, carried out using the computer language Magma, show, among other results, that none of these biplanes can be extended to a $3-(57,12,2)$ design. This had been suspected but never proved before; all the necessary arithmetic conditions for extendability are satisfied. Also, each of these biplanes is the only one to be found among the weight-11 vectors of its ternary code. All the coding-theoretic parameters of the five codes are computed as well as those of the sixteen non-isomorphic residual $2-(45,9,2)$ designs provided by the five biplanes. Sixteen non-isomorphic ternary codes are obtained.

A *spread* of a design, where the blocks are regarded as subsets of points, is a partition of the point set $\mathcal{P}$ into blocks. A *packing* or *parallelism* is a partition of the block set $\mathcal{B}$ into spreads. *Mathon* presents new algorithms for finding spreads and packings of sets with applications to combinatorial designs and finite geometries. An efficient deterministic method for spread enumeration is used to settle several existence problems for $t-$designs and partial geometries. Randomized algorithms based on tabu search are employed to construct new Steiner $5-$designs and large sets of combinatorial designs. In particular, partitions are found of the 4−subsets of a 16−set into 91 disjoint affine planes of order 4.

Steiner triple systems

A Pasch configuration in any incidence structure is a quadrilateral; that is, a set of six points and four blocks forming a quadrilateral. *Akbari*, *Khosrovshahi*, *Maysoori* and *Shariari* consider the size $f(n)$ of the largest set of triples on a set of size n containing no Pasch configuration. Upper and lower bounds for $f(n)$ found and applications of this result discussed.

Lindner, with many striking diagrams, leads us through the intricacies of embeddings for partial cycle systems. The embeddings of a partial 3−cycle system of order n in a 3−cycle system of order $3(2n+1)$ and of a partial 5−cycle system of order n in a 5−cycle system of order $5(2n+1)$ are carefully explained and both pictures and examples turn a difficult piece of design theory into an elegant survey. Embedding partial even-cycle systems is much easier than embedding partial odd-cycle systems and the complete proof is given of the fact that a partial m−cycle system of order n can be embedded in an m−cycle system of order $2mn+1$ when m is even.

A $2-(v,k,\lambda)$ design or *balanced incomplete block design* (BIBD) is is called *minci* when the set $\mathcal{B}$ of blocks of can be partitioned into as many as possible maximum parallel classes and one other parallel class; many special types of designs are of this type. In particular, a *Rosa triple system* is a minci Steiner triple system with $v \equiv 2 \pmod 3$. *Ling* and *Colbourn* settle the existence of these triple systems completely.

Difference sets

A $(v,k,\lambda)-$*difference set* in a group G of order v is a k-subset D of G, such that every element $g \neq 1$ of G has exactly λ representations $g = d_1 d_2^{-1}$ with $d_1, d_2 \in D$. As for designs, the parameter $n = k - \lambda$ is the *order* of the difference set. *Jungnickel* and *Schmidt* provide an update of the Jungnickel's 1992 survey on difference sets. Apart from reviewing the classical results, the authors describe new developments in the subject, some of whose techniques come from algebraic number theory and group characters, that have led to both the disproof of some long-standing conjectures and the construction of new families of difference sets. In recent years, research has focused on difference sets with $(v,n) > 1$, and all the new classes found are presented.

Latin squares

An intercalate in a Latin square of order n is a 2×2 subsquare. The maximum possible number is $\frac{1}{2}n^2(n-1)$. This is achieved for $n = 2^k$, but it is unknown what happens for other n. *Danziger* and *Mendelsohn* show, among other results, that Latin squares of order n such that every cell is contained in an intercalate exist when $n \neq 1, 3, 5, 7$.

Diagram geometries

A string diagram of type $c^* - c$ is a rank three geometry. It has three types of objects: points, lines, and blocks, which satisfy the following axioms:

(a) any two distinct lines incident with a common point p are both incident with a unique common block;

(b) any two lines incident with a common block are incident with a unique common point;

(c) given a point p incident with a block B, there are exactly two lines incident with both;

(d) (The String Property) any point incident with a line of a block B is also incident with B.

All examples fall into these six classes. *Shult* surveys the current situation.

Generalized quadrangles

Let S be a generalized quadrangle of order (s, t) and let p be a point of S. An *elation about* p is either the identity collineation or a collineation of S which fixes each line incident with p and fixes no point not collinear with p. If S admits a group G of elations about p acting regularly on the points not collinear with p then S is an *elation generalized quadrangle* with *elation group* G and *base point* p. *O'Keefe* and *Penttila* survey some recent classification theorems for elation generalized quadrangles of order (q^2, q), q even, with particular emphasis on those involving subquadrangles of order q.

If x is a regular point of a generalized quadrangle $\mathcal{S} = (P, B, \mathrm{I})$ of order $(s, t), s \neq 1$, then x defines a dual net with $t + 1$ points on any line and s lines through every point. The *Axiom of Veblen* states that if two lines intersect, then any two other lines meeting the first two and not passing through their meet also intersect. *Thas* and *Van Maldegehem* show that, if $s \neq t, s > 1, t > 1$, then $\mathcal{S}$ is isomorphic to a $T_3(O)$ of Tits if and only if $\mathcal{S}$ has a coregular point x such that, for each line L incident with x, the corresponding dual net satisfies the Axiom of Veblen. Translation generalized quadrangles are also considered.

Projective spaces

By considering the Veronese map from $PG(2, q)$ to $PG(5, q)$, *Bonisoli* and *Cossidente* construct a *cap*, that is a set of points no three of which are collinear, in $PG(5, q)$. It is still a mysterious problem to discover the largest size of cap in this space for $q > 3$.

A way of constructing a large *arc*, that is a set of points no three collinear, in $PG(2, q)$ with q a square is as an orbit of the $(q + \sqrt{q} + 1)$-th power of a Singer cycle. This produces a complete (maximal) arc of size $q - \sqrt{q} + 1$. For q even, this is as large as possible below the maximum size $q + 2$; for q odd, this is conjectured to be the next largest below the maximum size $q + 1$. In the even case, the associated algebraic curve has degree $\sqrt{q} + 1$ and is Hermitian; in the odd case, the curve has degree $2(\sqrt{q} + 1)$. In the latter case, *Cossidente* and *Korchmáros* show that the curve is birationally isomorphic to a Hermitian curve.

A *unital* is a $2 - (q^3 + 1, q + 1, 1)$ design, and a *Hermitian unital* is given by the rational points and non-tangent lines of a Hermitian curve. Buekenhout has given a construction of unitals in $PG(2, q^2)$ using the André representation of $PG(2, q^2)$ in the space $PG(4, q)$. Metz has shown that this construction produces Hermitian and non-Hermitian unitals. *Metsch* gives a geometric criterion in $PG(4, q)$ to decide whether the unital in $PG(2, q^2)$ is Hermitian or not.

A *spread* of $PG(3, q)$ is a set of $q^2 + 1$ lines which partition the points of the space. A *parallelism* or *packing* of $PG(3, q)$ is a collection of $q^2 + q + 1$ spreads which partition the lines of the space. A *regular* spread is one which contains the regulus determined by any three of its lines. A *subregular* spread of *index one* is a spread obtained from a regular spread by replacing a single regulus in the spread by its opposite. A *regular* parallelism is one consisting only of regular spreads. A *uniform* parallelism is one consisting entirely of spreads which are projectively equivalent. *Prince* shows that there are no regular parallelisms of $PG(3, 3)$ but that there are many parallelisms consisting entirely of subregular spreads of index one.

A double-five of planes is a set ψ of 35 points in $PG(5, 2)$ which admits two distinct decompositions $\psi = \alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4 \cup \alpha_5 = \beta_1 \cup \beta_2 \cup \beta_3 \cup \beta_4 \cup \beta_5$ into a set of five mutually skew planes such that $\alpha_r \cap \beta_r$ is a line, for each r, while $\alpha_r \cap \beta_s$ is a point, for $r \neq s$. *Shaw* delves into the properties of this surprisingly complex configuration. In particular the existence of an invariant symplectic form is demonstrated and some related duality properties are described.

On maximum size anti-Pasch sets of triples

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Abstract

Given an n-set X, we denote the cardinality of a maximum size anti-Pasch (Pasch free) set of triples of X by $f(n)$. In this paper we provide lower and upper bounds for $f(n)$ and consequently we disprove a conjecture posed by Khosrovshahi at the fifteenth British Combinatorial Conference (BCC15).

1 Introduction

Let $X = \{1, \cdots, n\}$. We denote by $P_i(X)$ the set of all subsets of X of size i, and the elements of $P_3(X)$ will be called *triples*. Let $P_c = \{A_1, A_2, A_3, A_4\} \subset P_3(X)$ be a collection of four triples of X. If the union of these triples has size 6 and if the intersection of each pair of distinct triples in P_c has size one then P_c is called a *Pasch configuration*. It is easy to see that $P_c = \{abc, axy, bxz, cyz\}$ for some choice of $a, b, c, x, y, z \in X$. Thus a Pasch is exactly a Fano plane minus one point and the three lines incident with that point. A subset of $P_3(X)$ that does not contain a Pasch is called an *anti-Pasch*.

We denote by $f(n)$ the cardinality of an anti-Pasch subset of $P_3(X)$ with maximum size. At the 15th British Combinatorial Conference, it was conjectured by Khosrovshahi [7] that $f(n) \leq \binom{n}{2}$. In this paper we present a counterexample to this conjecture. We will show that for $n \geq 7$

$$\frac{5}{4}(\binom{n}{2} - n) \leq f(n) \leq \frac{4}{7}\binom{n}{3}.$$

The lower bound disproves Khosrovshahi's Conjecture, and the very crude upper bound will help determine the exact value of $f(n)$ for small values of n. In fact, we will show that $f(6) = 14$ and $f(7) = 20$.

The existence of anti-Pasch configurations with additional properties has been the object of much study. Recall that a *Steiner triple system* of order v is a collection of triples on a set with v elements such that each unordered pair of elements is contained in exactly one triple in the collection. It has been conjectured [1] that if $v \cong 1$ or $3 \pmod 6$ and $v \neq 7, 13$ then an anti-Pasch Steiner triple system exists. Even though the conjecture remains open, infinite families of such systems have been found [1, 3, 5, 6, 9, 10]. The above conjecture is the refinement of a special case of a more general conjecture of Paul Erdös. Let $r \geq 3$ be an integer. A Steiner triple system is called *r-sparse* if every set of $r+2$ points carries fewer than r triples. It is easy to see that a Steiner triple system is 4-sparse if and only if it is anti-Pasch, and an Steiner triple system is $r-1$-sparse if it is r-sparse. Erdös [4] (see also [1, 2, 8]) asks whether for any $r \geq 3$ there is a v_r such that for all $v > v_r$ and $v \equiv 1$ or $3 \pmod 6$ there exists an r-sparse Steiner triple system on v points. For an infinite family of 5-sparse Steiner triple systems as well as related results see [2]. For $r > 5$ very little is known about this seemingly very difficult question.

We conclude the introduction with some motivation for the conjecture of Khosrovshahi [7]. We construct an $\binom{n}{2} \times \binom{n}{3}$ matrix, $W(n)$, of zeros and ones with the rows indexed by the elements of $P_2(X)$, and the columns indexed by the elements of $P_3(X)$. If $A \in P_2(X)$ and $B \in P_3(X)$ then the (A, B) entry of the matrix is defined to be:

$$W(n)(A,B) = \begin{cases} 1 & \text{if } A \subseteq B, \\ 0 & \text{otherwise.} \end{cases}$$

It is known [4] that

$$\operatorname{rank} W(n) = \binom{n}{2}.$$

Therefore, any $\binom{n}{2} + 1$ columns of $W(n)$ must be linearly dependent.

Let $P_c = \{abc, axy, bxz, cyz\}$ be a Pasch configuration in $P_3(X)$ and define

$$\overline{P}_c = \{xyz, bcz, acy, abx\}.$$

$\overline{P}_c$ is also a Pasch configuration and is called the counterpart of P_c. Now given any Pasch configuration P_c, we define a $(0, 1, -1)$-vector

$$f = \left(f_1, \cdots, f_{\binom{n}{3}}\right)$$

as follows:

$$f_i = \begin{cases} 1 & \text{if } B_i \in P_c, \\ -1 & \text{if } B_i \in \overline{P}_c, \\ 0 & \text{otherwise,} \end{cases}$$

where B_i is the i-th indexed element of $P_3(X)$. It is very easy to check that $f \in \ker W(n)$. Since the linear dependence of any collection of the columns of $W(n)$ gives an element of the kernel, it was thus conjectured that any $\left(\binom{n}{2} + 1\right)$-subset of $P_3(X)$ would contain a Pasch and its counterpart. As was noted above our example will show that this conjecture is false.

2 Preliminaries

In this section we gather some further notation, as well as definitions, and some known simple facts.

Let $X = \{1, 2, \ldots, n\}$. For any $x \in X$, we denote by $x**$, the set of all elements of $P_3(X)$ which contain x. Similarly, for any $\{x, y\} \in P_2(X)$, $xy*$ is the collection of triples that contain x and y. For any $k, 1 \leq k \leq n$, and $B \in P_k(X)$,

$$\Delta B = P_{k-1}(B)$$

is called the *shadow* of B.

Let $[7] = \{1, \cdots, 7\}$, then the set

$$\mathcal{F} = \{B_1, \cdots, B_7\} \subset P_3([7])$$

is called a *Fano plane* if for any $B_i, B_j \in \mathcal{F}$ $(i \neq j)$, we have $|B_i \cap B_j| = 1$. It is known that the structure of a Fano plane is unique up to isomorphism [11] and is of the following form:

$$\{123, 145, 167, 246, 257, 347, 356\}.$$

It is also easy to show that there exist 30 distinct Fano planes on 7 points and for any $B \in P_3([7])$, there exist exactly 6 Fano planes containing B. It is straightforward to see that every 5 triples of a Fano plane contain a Pasch.

Now, suppose $n \geq 6$ and $X = \{1, \cdots, n\}$. Then we have the following easily verified facts [11]:

(*i*) For any $B \in P_3(X)$, there exist $(n-3)(n-4)(n-5)$ Pasches containing B;

(*ii*) There are all together $30\binom{n}{6}$ Pasches in $P_3(X)$;

(*iii*) Let $A, B \in P_3(X)$, with $|A \cap B| = 1$, then there are $2(n-5)$ Pasches containing both A and B.

3 Lower and Upper Bounds for $f(n)$

Recall that $f(n)$ is the maximum size of an anti-Pasch subset of $P_3(X)$. In this section we obtain some bounds for $f(n)$.

Theorem 1 *Let $n \geq 3$ be a positive integer, and let l be any integer with $1 \leq l \leq \lfloor \frac{n-1}{2} \rfloor$. Let $X = \{1, \cdots, n\}$, and define*

$$\begin{aligned} A_0 &= 1**, \\ A_i &= (2i)(2i+1)*\setminus A_0, \quad \textit{for} \quad i = 1, 2, \cdots, l. \end{aligned}$$

Let $\mathcal{A} = \cup_{i=0}^{l} A_i$. *Define* $\mathcal{E} = \cup_{k=2}^{l} \cup_{j=1}^{k-1} E_{jk}$, *where*

$$E_{jk} = \{1(2j)(2k), 1(2j)(2k+1), 1(2j+1)(2k), 1(2j+1)(2k+1)\},$$

for $1 \leq j < k \leq l$. *Then* $\mathcal{A} \backslash \mathcal{E}$ *is an anti-Pasch set with* $\binom{n-1}{2} + l(n-3) - 4\binom{l}{2}$ *elements.*

Example. For $n = 9$ and $l = 2$ the above construction gives

$$\begin{aligned} \mathcal{A} &= \cup_{i=0}^{2} \mathcal{A}_i = 1**\cup 23*\cup 45*; \\ \mathcal{E} &= \{124, 125, 134, 135\}. \end{aligned}$$

Thus $\mathcal{A} \backslash \mathcal{E}$ is an anti-Pasch set with $28 + 6 + 6 - 4 = 36$ triples.

Proof In a Pasch in $P_3(X)$ every element of X either does not occur or it occurs exactly twice. Likewise every pair of elements of X either are not contained in any of the triples or they are contained in exactly one of the triples. Thus if $\mathcal{A} \backslash \mathcal{E}$ contained a Pasch, this Pasch could have either zero or two triples from A_0 and at most one triple from any of the A_i for $i = 1, \ldots, l$. In addition, this Pasch could not have elements from four different A_i's since this would lead to the presence of at least 7 elements of X in the Pasch, and every Pasch uses only 6 of the elements of X. This leaves only the possibility that the Pasch would have two triples from A_0 and one triple from A_j and one from A_k where $1 \leq j < k \leq l$. Thus the elements of X that are contained in the triples of the Pasch are 1, $2j$, $2j+1$, $2k$, $2k+1$, and a sixth element a. Now the pair $1a$ can occur in at most one triple and thus at least one of the triples from A_0 will not contain a. Thus this triple will consist of 1, one element of $\{2j, 2j+1\}$ and one element of $\{2k, 2k+1\}$. This means that one of the triples has to come from E_{jk}, and these triples were eliminated in $\mathcal{A} \backslash \mathcal{E}$. Thus $\mathcal{A} \backslash \mathcal{E}$ is an anti-Pasch configuration. Clearly,

$$\begin{aligned} |\mathcal{A}_0| &= \binom{n-1}{2}, \\ |\mathcal{A}_i| &= n-3, \quad i = 1, 2, \cdots, l, \\ |\mathcal{E}| &= 4\binom{l}{2}. \end{aligned}$$

Therefore the size of $\mathcal{A} \backslash \mathcal{E}$ is as claimed and the proof is complete. □

To find a large anti-Pasch configuration we choose l so as to maximize $|\mathcal{A} \backslash \mathcal{E}|$. For a fixed n the expression for $|\mathcal{A} \backslash \mathcal{E}|$ is a second degree polynomial in l with its maximum at $l = \frac{n-1}{4}$. Choosing $l = \lfloor \frac{n}{4} \rfloor$ (which is the closest integer to $\frac{n-1}{4}$) gives the largest possible value for $|\mathcal{A} \backslash \mathcal{E}|$ for an integer l. A straightforward calculation shows that, for $l = \lfloor \frac{n}{4} \rfloor$, we have $|\mathcal{A} \backslash \mathcal{E}| \geq$

$\frac{5n^2-14n+5}{8}$ and that this expression is in turn no smaller than $\frac{5}{4}\left(\binom{n}{2}-n\right)$. Thus we have proved:

Corollary 2 *Let X be a set of size n with $n \geq 3$. Let $f(n)$ be the maximum size of an anti-Pasch subset of $P_3(X)$. Then*

$$f(n) \geq \frac{5n^2-14n+5}{8} \geq \frac{5}{4}(\binom{n}{2}-n). \qquad \square$$

It follows that for $n \geq 10$ we have an anti-Pasch configuration in $P_3(X)$ of size greater or equal to $\binom{n}{2}+1$. This clearly provides a counterexample to Khosrovshahi's Conjecture. We now give a crude upper bound for $f(n)$.

Theorem 3 *Let X be a set of size n with $n \geq 7$, and let $f(n)$ be the maximum size of an anti-Pasch subset of $P_3(X)$. Then*

$$f(n) \leq \frac{4}{7}\binom{n}{3}.$$

Proof Suppose $\mathcal{A}$ is an anti-Pasch set of size $f(n)$, and let $\overline{\mathcal{A}}$ denote $P_3(X)\backslash\mathcal{A}$. Now, let

$$\Sigma = \{(B,\mathcal{F})|B \in \mathcal{F} \cap \overline{\mathcal{A}}, \mathcal{F} \text{ is a Fano plane in } P_3(X)\}.$$

We will count the elements of Σ in two ways. Note that there exists $\binom{n}{7} \times 30$ distinct Fano planes in $P_3(X)$. Let $\mathcal{F}$ be an arbitrary Fano plane in $P_3(X)$. Every five triples of a Fano plane contain a Pasch and thus since $\mathcal{A}$ is an anti-Pasch set we must have $|\mathcal{F} \cap \overline{\mathcal{A}}| \geq 3$. Thus

$$|\Sigma| \geq \binom{n}{7} \times 30 \times 3.$$

On the other hand, let $B \in \overline{\mathcal{A}}$ be an arbitrary triple, then there are $\binom{n-3}{4} \times 6$ distinct Fano planes containing B. Since $|\overline{\mathcal{A}}| = |P_3(X)\backslash\mathcal{A}| = \binom{n}{3} - f(n)$, it follows that

$$|\Sigma| = (\binom{n}{3} - f(n))\binom{n-3}{4} \times 6.$$

Solving for $f(n)$ we get $f(n) \leq \frac{4}{7}\binom{n}{3}$. $\square$

Proposition 4 *Let $g(n) = f(n)/\binom{n}{3}$. Then $g(n)$ is a non-increasing function. In particular, if there exists a real number c and a natural number n_0 such that $f(n_0) \leq c\binom{n_0}{3}$, then $f(n) \leq c\binom{n}{3}$ for any $n \geq n_0$.*