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Equilibrium States in Ergodic Theory

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Preface

This book grew out of a graduate course on equilibrium states in ergodic theory which I gave twice (in winter 1994/95 and 1996/97) at the University of Erlangen. The text, as it is now, covers rather exactly 32 lectures of 90 minutes each plus a number of exercises. As my intention was (and still is) to put no more into this book than I was able to teach in such a one semester course, the material which I finally included strongly reflects my personal view of the field and is limited in both breadth and depth. But where the knowledgeable reader may wonder why certain topics are omitted or treated only under restrictive assumptions, the student who reads this book will hopefully profit from the concentration on key concepts and key examples gained in this way.

My goal in writing this book was to provide an introduction to the ergodic theory of equilibrium states which gives equal weight to two of its most important applications, namely to equilibrium statistical mechanics and to (time discrete) dynamical systems. In selecting the material, I always kept in mind some of the prime examples of these two fields: the two-dimensional Ising model, piecewise differentiable maps of an interval, and conformal iterated function systems. After working through the book, the reader not only has a solid basic knowledge of the general theory of equilibrium states, but has also seen applications of the theory to such different concepts as phase transitions, observable measures, and the dimension of fractals.

The book starts with a chapter on equilibrium states on finite probability spaces which motivates most of the theory developed later. In this setting, equilibrium states are just probability vectors maximizing a quantity of the type "entropy + energy", i.e., they are distinguished by means of variational principles. Gibbs measures, large deviation estimates, the Ising model on a finite lattice, equilibrium states adapted to Markovian transition mechanisms and absolutely continuous invariant measures are discussed in a nontechnical and completely elementary setting.

In the second chapter, the measure theoretic framework for the general theory is set up: measure preserving $\mathbb{Z}^d$- or $\mathbb{Z}_+^d$-actions on probability spaces. Besides the ergodic theorem the ergodic decomposition is also proved for this class of systems. An extra chapter is devoted to entropy theory for measure preserving actions. It provides a number of tools for entropy calculations needed later, most notably the Shannon–McMillan–Breiman theorem (for $d = 1$) and the Kolmogorov–Sinai theorem.

Chapter 4 on equilibrium states and pressure forms the core of the book. The pressure $p(\phi)$ of a continuous function ϕ is defined from a variational point of view, and the identification of the pressure as a quantity that can be approximated by evaluating ϕ at finitely many points (the "variational principle") is postponed to Section 4.4. The proof of this result is organized in such a way that in certain cases equilibrium states are also constructively approximated. But before this is done, continuity properties of the entropy function and the convex geometric meaning of equilibrium states as derivatives of the pressure function $\phi \mapsto p(\phi)$ are elucidated. Two aspects of this chapter may be particularly noteworthy: (1) Nearly all results apply not only to continuous ϕ but also to upper semicontinuous ones. This allows us e.g., to include interval maps with derivative $+\infty$ in the considerations of the last chapter. (2) The convex geometric point of view makes it possible to prove for expansive actions in Theorem 4.5.9 that the set of all equilibrium states for a given ϕ is the closed convex hull of the set of those equilibrium states that are constructed in the course of the proof of the variational principle.

Chapter 5 is devoted to Gibbs measures. Although this notion has proved very useful for smooth dynamical systems also, we concentrate here on the statistical mechanical setting and restrict to shift spaces over the d-dimensional integer lattice. Gibbs measures for regular functions ϕ are introduced and, using some of the convex geometric results from Chapter 4, they are identified as equilibrium states and vice versa. Furthermore, a large deviations principle for Gibbs measures is proved, stationary measures for finite state Markov chains are identified as Gibbs measures, and the phase transition for the two-dimensional Ising model is investigated.

The last chapter deals with (piecewise) differentiable dynamical systems, in particular with piecewise monotonic interval maps and with iterated function systems. Observability and absolute continuity (w.r.t. Lebesgue measure) of invariant measures are related to their variational properties and simple versions of Rohlin's formula and Ruelle's inequality are derived. For conformal iterated function systems invariant measures of maximal dimension are identified as certain equilibrium states.

The appendix collects a number of facts from analysis, measure theory and probability theory used throughout the book. They are given without proofs but with references to other textbooks.

It is a pleasure for me to thank the following students and colleagues for many helpful remarks and suggestions: Henk Bruin, Bernhard Burgeth, Achim Klenke, Matthias St Pierre, Karl Straußberger, Jan Wenzelburger, Roland Zweimüller. The figures showing equilibrium states of the Ising model at various temperatures were contributed by Achim Klenke. Although I did not follow any existing text, the courses (and hence these notes) were certainly influenced by many books from

which I learned ergodic theory. They are all contained in the bibliography.

The index at the end of the book is preceded by a list of special notations which hopefully will help to settle most notational problems. Standard mathematical symbols do not need extra mentioning except perhaps for the conventions that $\mathbb{N} = \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ and that $O(f(n))$ and $o(f(n))$ may stand for any term $g(n)$ such that $\sup_{n\in\mathbb{N}} \frac{|g(n)|}{|f(n)|} < \infty$ or $\lim_{n\to\infty} \frac{|g(n)|}{|f(n)|} = 0$, respectively.

Erlangen, May 1997,
Gerhard Keller