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Excerpt
More information

1. Algebras

We will assume that all of our rings have an identity. If R is a ring, an abelian group M is a **left R-module** if for every $r \in R$ and $m \in M$, there is a unique element $rm \in M$ such that

$$r(a+b) = ra + rb\,,$$

$$(r+s)a = ra + sa\,,$$

$$(rs)a = r(sa)\,,$$

$$1_R a = a$$

for all $r, s \in R$ and $a, b \in M$. In the same way, but multiplying on the right, we define a **right R-module**.

If M and N are left R-modules, a map $f : M \to N$ is **R-linear** if f is additive and $f(rm) = rf(m)$ for all $m \in M$ and $r \in R$.

(1.1) DEFINITION. Suppose that R is a commutative ring and suppose that A is a left R-module. If A also is a ring such that

$$(ra)b = r(ab) = a(rb)$$

for all $r \in R$ and all $a, b \in A$, we say that A is an **R-algebra**.

When we think of R-algebras, we have two important examples in mind: $\mathrm{Mat}(n, R)$, the R-algebra of $n \times n$ matrices with entries in R and, for every finite group G, the **group algebra**

$$RG = \{\sum_{g\in G} a_g g \,|\, a_g \in R\}$$

with the multiplication of G extended linearly to RG. (In fact, representation theory studies the homomorphisms between RG and $\mathrm{Mat}(n, R)$.)

Also, if A is any ring and R is a subring of $\mathbf{Z}(A) = \{a \in A \mid ax = xa$ for all $x \in A\}$ (with the identity of A inside R), then A is an R-algebra.

If A and B are R-algebras, an **algebra homomorphism** is an R-linear, multiplicative map $f : A \to B$ such that $f(1_A) = 1_B$.

For the rest of this chapter, A is an R-algebra.

(1.2) DEFINITION. A left R-module V is said to be an A-**module** if V is a right A-module (A considered as a ring) such that for all $v \in V$, $r \in R$ and $a \in A$, we have that

$$(rv)a = r(va) = v(ra)\,.$$

One of the most important examples of an A-module is A itself with right multiplication. This is usually called the **regular** A-module.

If V is an A-module, a subgroup W of V is an A**-submodule** if $wa \in W$ for all $w \in W$ and $a \in A$. Notice that A-submodules are necessarily R-submodules since $rv = v(r1_A)$ for $r \in R$ and $v \in V$. Observe that the A-submodules of the regular A-module are the right ideals of A.

If W is an A-submodule of V, then V/W is an A-module via

$$(v + W)a = va + W$$

for $v \in V$ and $a \in A$.

(1.3) DEFINITION. We say that a nonzero A-module V is **simple** if its only A-submodules are 0 and V. (It is also common to say, in this case, that V is **irreducible**.)

If V and W are A-modules, an additive map $f : V \to W$ such that

$$f(va) = f(v)a$$

for all $v \in V$ and $a \in A$ is an A**-homomorphism** of modules. A bijective A-homomorphism is an **isomorphism** and we write $V \cong W$ in this case.

Notice that A-homomorphisms are necessarily R-linear since $f(rv) = f(v(r1_A)) = f(v)(r1_A) = rf(v)$ for $r \in R$ and $v \in V$.

If $f : V \to W$ is an A-homomorphism, then $\ker(f) = \{v \in V \mid f(v) = 0\}$ and $\mathrm{Im}(f)$ are A-submodules of V and W, respectively. Also, the map $v + \ker(f) \mapsto f(v)$ defines an isomorphism $V/\ker(f) \cong \mathrm{Im}(f)$.

If V and W are A-modules, we write $\mathrm{Hom}_A(V, W)$ for the abelian group of all A-homomorphisms $V \to W$. If $r \in R$ and $f \in \mathrm{Hom}_A(V, W)$, then $\mathrm{Hom}_A(V, W)$ is a left R-module via $(rf)(v) = rf(v)$ for $v \in V$. The set of all A-homomorphisms $V \to V$ is denoted by $\mathrm{End}_A(V)$. It is easy to check that $\mathrm{End}_A(V)$ is an R-algebra. Furthermore, R and $R1_V = \{r1_V \mid r \in R\}$ may be identified whenever R is a field.

(1.4) LEMMA (Schur). *Suppose that V and W are simple A-modules. Then every nonzero A-homomorphism $f : V \to W$ is invertible. As a consequence, if R is an algebraically closed field and $\dim_F(V)$ is finite, then $\mathrm{End}_A(V) = R$.*

Proof. Suppose that $f : V \to W$ is nonzero. Since $\ker(f)$ and $\mathrm{Im}(f)$ are A-submodules of V and W, respectively, it follows that $\ker(f) = 0$ and $\mathrm{Im}(f) = W$. Then f is bijective. To prove the latter assertion, we choose $0 \neq \lambda \in R$, an eigenvalue of f. Then $f - \lambda 1_V \in \mathrm{End}_A(V)$ is not invertible and therefore $f = \lambda 1_V$, by applying the first part. ∎

Sometimes, we use the fact that if R is an algebraically closed field and $f : V \to W$ is a nonzero A-homomorphism between two simple finite dimensional A-modules, then $\mathrm{Hom}_A(V, W) = \{rf \mid r \in R\}$. This easily follows from the second part in Schur's lemma.

If I is an ideal (we always mean double sided) of A, it is straightforward to check that A/I is an R-algebra.

The **annihilator** of an A-module V is $\mathrm{ann}(V) = \{a \in A \mid va = 0$ for all $v \in V\}$. This is an ideal of A, and notice that we may view V as an $(A/\mathrm{ann}(V))$-module.

We define the **Jacobson radical** of an R-algebra A to be the intersection of all $\mathrm{ann}(V)$ where V runs over all the simple A-modules. It is denoted by $\mathbf{J}(A)$, and certainly it is an ideal of A.

The next result tells us where to find the simple A-modules.

(1.5) THEOREM. *If A is an R-algebra and V is a simple A-module, then there exists a maximal right ideal I of A such that V and A/I are isomorphic. In fact, $\mathbf{J}(A)$ is the intersection of all maximal right ideals of A.*

Proof. If $0 \neq v \in V$, then $v \in vA = \{va \mid a \in A\}$. Thus, $vA = V$ since vA is a nonzero A-submodule of V. Now, the map $a \mapsto va$ from A onto V is an A-homomorphism of A-modules. Since $I = \mathrm{ann}(v) = \{a \in A \mid va = 0\}$ is the kernel of the map, A/I is isomorphic to V. The fact that V is simple makes I a maximal right ideal of A. Now, if J is the intersection of all maximal right ideals of A, we have that

$$J \subseteq \bigcap_{v \in V} \mathrm{ann}(v) = \mathrm{ann}(V)$$

and thus $\mathbf{J}(A)$ contains J. Now, if L is any maximal right ideal, then A/L is a simple A-module. Also, $\mathrm{ann}(A/L) \subseteq \mathrm{ann}(1 + L) = L$. Hence, $\mathbf{J}(A) \subseteq L$ for all such L. Thus $\mathbf{J}(A) \subseteq J$ and the proof of the theorem is completed. ∎

If $\mathbf{J}(A)$ is the unique maximal ideal of A, we say that A is **local**.

There is a useful fact about the elements of the Jacobson radical which will be used later on.

(1.6) THEOREM. *If A is an R-algebra and $a \in \mathbf{J}(A)$, then $1 - a$ is invertible.*

Proof. If $(1-a)A < A$, then the right ideal $(1-a)A$ is contained in some maximal right ideal M of A. In this case, since $\mathbf{J}(A) \subseteq M$, we have that $a \in M$ and we conclude that $1 \in M$, a contradiction. Therefore, we see that $(1-a)A = A$. Thus, we may find $1-b \in A$ such that $(1-a)(1-b) = 1$. We just need to prove that $(1-b)(1-a) = 1$. Since $(1-a)(1-b) = 1$, we see that $b = a(b-1) \in \mathbf{J}(A)$. Hence, by the same reasoning as before, $1-b$ has a right inverse, say c. Now, $1 - a = (1-a)((1-b)c) = ((1-a)(1-b))c = c$ and therefore $1-b$ is a left and right inverse of $1-a$, as required. ∎

If V is an A-module, $W \subseteq V$ and I is a right ideal of A, then WI denotes the additive subgroup of V generated by all the products wx with $w \in W$ and $x \in I$. Notice that WI is an A-submodule of V. By repeated application of this definition (with $V = A$), we can define I^n for every positive integer n. The right ideal I is **nilpotent** if there is an n with $I^n = 0$. (Note that $I^n = 0$ if and only if every product of n elements of I is zero.)

An A-module V is **finitely generated** if there exist $v_1, \ldots, v_n \in V$ such that

$$V = v_1 A + \ldots + v_n A\,.$$

(1.7) LEMMA (Nakayama). *Suppose that W is an A-submodule of V such that V/W is finitely generated over A. If $V = W + V\mathbf{J}(A)$, then $V = W$.*

Proof. It suffices to show the lemma for the case $W = 0$ and afterwards to apply it to V/W. So we have that V is a finitely generated A-module such that $V\mathbf{J}(A) = V$ and we wish to prove that $V = 0$. If $V \neq 0$, let $X \neq \emptyset$ be a minimal A-generating subset of V. Now,

$$V = V\mathbf{J}(A) = (\sum_{x \in X} xA)\mathbf{J}(A) = \sum_{x \in X} x\mathbf{J}(A)\,.$$

If $y \in X$, then we may write

$$y = \sum_{x \in X} x a_x\,,$$

where $a_x \in \mathbf{J}(A)$. Now,

$$y(1 - a_y) = \sum_{x \in X - \{y\}} x a_x$$

and thus, by applying Theorem (1.6), we have that

$$y = \sum_{x \in X - \{y\}} x a_x (1 - a_y)^{-1}\,.$$

Therefore, $X - \{y\}$ generates V over A which contradicts the minimality of X. ∎

From now until the end of this chapter, we assume that $R = F$ is a field. Hence, from now on, every F-algebra is a vector space over F. We will assume that A and, in general, every A-module have finite dimension over F. Notice that, in this case, we may also assume that $F \subseteq \mathbf{Z}(A)$ since the map $f \mapsto f1_A$ is an injective ring homomorphism from F into $\mathbf{Z}(A)$.

(1.8) THEOREM. *Suppose that A is an F-algebra. Then $\mathbf{J}(A)$ is the unique maximal nilpotent right ideal of A. Moreover,*

$$\mathbf{J}(\mathbf{Z}(A)) = \mathbf{J}(A) \cap \mathbf{Z}(A)\,.$$

Proof. We have that $\mathbf{J}(A)^n$ is an F-subspace of A, and thus it is finitely generated over A (since A contains F). By Nakayama's lemma (1.7), we have that $\mathbf{J}(A)^{n+1}$ is smaller than $\mathbf{J}(A)^n$, if this is nonzero. Hence, since the dimension of A is finite, we see that $\mathbf{J}(A)$ is necessarily nilpotent. Now, if I is a nilpotent right ideal of A and V is a simple A-module, then $VI = 0$ or $VI = V$ since VI is an A-submodule of V. If $VI = V$, then $VI^2 = (VI)I = VI = V$ and, in general, $VI^n = V$. But this is impossible because there is an integer m with $I^m = 0$. Thus, $VI = 0$ for all simple A-modules V and hence, $I \subseteq \mathbf{J}(A)$, as desired.

Finally, $\mathbf{J}(A) \cap \mathbf{Z}(A)$ is a nilpotent ideal of $\mathbf{Z}(A)$ and therefore, by the first part, it is contained in $\mathbf{J}(\mathbf{Z}(A))$. Now, let $z \in \mathbf{J}(\mathbf{Z}(A))$. Since $\mathbf{J}(\mathbf{Z}(A))$ is nilpotent and commutes with the elements of A, we have that zA is a nilpotent right ideal of A. Then $z \in \mathbf{J}(A)$. This proves that $\mathbf{J}(\mathbf{Z}(A)) \subseteq \mathbf{J}(A) \cap \mathbf{Z}(A)$. ∎

(1.9) DEFINITION. If A is an F-algebra, we say that A is **semisimple** if $\mathbf{J}(A) = 0$. Also, we say that A is **simple** if it has no proper (two sided) ideals.

Since A and $A/\mathbf{J}(A)$ have the same set of simple A-modules, it follows that $A/\mathbf{J}(A)$ is semisimple.

An A-module V is said to be **completely reducible** if it is the direct sum of simple A-submodules. (It is also common in this case to say that V is **semisimple**.) In fact, there is no difference between completely reducible modules and modules which may be written as a sum (not necessarily direct) of simple submodules.

(1.10) LEMMA. *Let V be an A-module and suppose $V = \sum V_i$, where the V_i's are simple submodules. Then V is the direct sum of some of the V_i's.*

Proof. Since V has finite dimension, we let W be an A-submodule of V maximal with respect to the property that W is the direct sum of some of the V_i's. If W is proper, then there exists a V_j not contained in W. But then, since V_j is simple, we have that $V_j \cap W = 0$. Then $W + V_j$ is a direct sum, which contradicts the maximality of W. ∎

More interesting is the next result.

(1.11) THEOREM. *If A is an F-algebra and V is an A-module, then the following conditions are equivalent.*

(a) V is completely reducible.

(b) If U is an A-submodule of V, then there is an A-submodule W such that $V = U \oplus W$.

Proof. Write $V = \sum V_i$, where the V_i's are simple submodules, and suppose that U is an A-submodule of V. Since V has finite dimension, let W be an A-submodule of V maximal such that $U + W = U \oplus W$. If $U + W$ is proper, then there is some V_j not contained in $U + W$. Since V_j is simple, $V_j \cap (U + W) = 0$. Therefore, $U + (V_j + W) = U \oplus (V_j + W)$, which contradicts the maximality of W. This proves that (a) implies (b).

Assume (b) and, since V is finite dimensional, let U be an A-submodule of V maximal such that U is a sum of simple A-submodules. By hypothesis, there is an A-submodule W of V such that $V = U \oplus W$. If $W \neq 0$, since V is finite dimensional, we may find W_0, a simple submodule of V inside W. Then $U + W_0 > U$, which contradicts the maximality of U. Hence, $U = V$ is completely reducible. ∎

(1.12) COROLLARY. *Suppose that V is a completely reducible A-module. If U is an A-submodule of V, then U and V/U are completely reducible.*

Proof. By Theorem (1.11), we have that V/U is isomorphic to a submodule of V. Hence, it suffices to show the first part. If W is an A-submodule of U, again by Theorem (1.11) we know that there exists an A-submodule W_0 of V such that $V = W \oplus W_0$. Then $U = W \oplus (U \cap W_0)$. This proves that U is completely reducible. ∎

If $V_1, \ldots, V_n$ are A-modules, we may form the **external direct sum** of $V_1, \ldots, V_n$, which is denoted by $V_1 \oplus \ldots \oplus V_n$, by setting $V_1 \oplus \ldots \oplus V_n = V_1 \times \ldots \times V_n$ with the action

$$(v_1, \ldots, v_n)a = (v_1 a, \ldots, v_n a)$$

for $v_i \in V_i$ and $a \in A$. It is clear that if V_i is a simple A-module for all i, then $V_1 \oplus \ldots \oplus V_n$ is a completely reducible A-module.

(1.13) THEOREM. *Suppose that A is an F-algebra. Then A is semisimple if and only if every A-module is completely reducible.*

Proof. Assume first that every A-module is completely reducible. If we consider A, the regular A-module, by hypothesis we have that $A = \sum_i I_i$ is a sum of minimal right ideals. Hence, $\mathbf{J}(A) = A\mathbf{J}(A) = 0$ since $\mathbf{J}(A)$ annihilates the simple A-module I_i for all i.

Assume now that A is semisimple. First, we prove that the regular A-module A is completely reducible. To do that, we claim that there exist maximal right ideals $M_1, \ldots, M_n$ of A such that

$$\bigcap_{j=1}^{n} M_j = 0\,.$$

If this is the case, the map $a \mapsto (a+M_1, \ldots, a+M_n)$ maps A isomorphically into a submodule of the completely reducible A-module $A/M_1 \oplus \ldots \oplus A/M_n$. Then, by Corollary (1.12), A is completely reducible. To prove the claim, among the subspaces L of A which are intersections of a finite number of maximal right ideals, we choose L of minimal dimension. If $L \neq 0$, then L is not contained in $\mathbf{J}(A) = 0$. Since $\mathbf{J}(A)$ is the intersection of all the maximal right ideals of A (Theorem (1.5)), we have that there exists a maximal right ideal M such that $L \cap M < M$. This contradicts the choice of L and proves the claim.

Now, write $A = \sum_i I_i$ as a sum of minimal right ideals of A. If V is an A-module and $\mathcal{B}$ is an F-basis of V, we have that

$$V = \sum_{v \in \mathcal{B}, i} vI_i\,.$$

Since the map $I_i \to vI_i$ given by $x \mapsto vx$ is a surjective A-homomorphism and I_i is a minimal right ideal, it follows that the kernel of the map is I_i or zero. Hence, vI_i is isomorphic to I_i or 0. Therefore, V is a sum of simple A-submodules, as required. ∎

(1.14) COROLLARY. *If A is a semisimple F-algebra and B is an ideal of A, then the F-algebra A/B is semisimple.*

Proof. If V is an (A/B)-module, then V is an A-module with $va = v(a+B)$ for $v \in V$ and $a \in A$. Hence, V is a sum of simple A-submodules. Since $VB = 0$, these are also simple (A/B)-submodules of V. Now, Theorem (1.13) applies. ∎

(1.15) COROLLARY. *If A is a semisimple F-algebra, then every simple A-module is isomorphic to a minimal right ideal of A.*

Proof. If V is a simple A-module, by Theorem (1.5) we know that V is isomorphic to A/I, where I is a maximal right ideal of A. Since A is completely reducible, by Theorem (1.11) there exists a right ideal J such that $A = I \oplus J$. Since $J \cong A/I$, we have that J is a minimal right ideal and the result follows. ∎

(A word on notation is appropriate here. Although we usually write functions on the left, the notation fg indicates that f is applied first and then g.)

If $a \in A$ and V is an A-module, we denote by a_V the right multiplication map $V \to V$ given by $v \mapsto va$. It is easy to check that the map $a \mapsto a_V$ is an algebra homomorphism $A \to \mathrm{End}_F(V)$. Furthermore, the kernel of this map is $\mathrm{ann}(V)$. We also denote its image by $A_V = \{a_V \mid a \in A\}$.

(1.16) THEOREM (Double Centralizer). *Suppose that A is a simple F-algebra and let $I \neq 0$ be a right ideal of A. If $B = \mathrm{End}_A(I)$, then I is a B-module and the natural map $A \to \mathrm{End}_B(I)$ given by $a \mapsto a_I$ is an isomorphism of F-algebras. Consequently, if I is a minimal right ideal of A and F is algebraically closed, then A is isomorphic to $\mathrm{End}_F(I)$.*

Proof (Rieffel). As we said before Schur's lemma, it is straightforward to check that B is an F-algebra. Now, I is a B-module with $xf = f(x)$, for $x \in I$ and $f \in B$. Hence, $\mathrm{End}_B(I)$ is again an F-algebra.

First notice that $a_I \in \mathrm{End}_B(I)$ because if $x \in I$ and $f \in B$, we have that

$$a_I(xf) = (xf)a = f(x)a = f(xa) = a_I(x)f\,.$$

We wish to prove that the map $a \mapsto a_I$ is an algebra isomorphism. By the comments preceding the statement of this theorem, we have that the map $a \mapsto a_I$ is an algebra homomorphism with kernel $\mathrm{ann}(I)$. Since $\mathrm{ann}(I)$ is a proper two sided ideal of the simple algebra A, we conclude that our map is injective.

What we need to prove is that $A_I = \{a_I \mid a \in A\} = \mathrm{End}_B(I)$. Since the identity lies in A_I, it suffices to show that A_I is a right ideal of the ring $\mathrm{End}_B(I)$. Since $AI = A$ (because $AI \neq 0$ is a two sided ideal of A), it suffices to show that $(AI)_I$ is a right ideal of $\mathrm{End}_B(I)$. Let $a \in A$, $y \in I$ and $f \in \mathrm{End}_B(I)$. We prove that $(ay)_I f \in \mathrm{End}_B(I)$. If $u \in I$, let us denote by $u_l : I \to I$ the map given by $u_l(x) = ux$. Notice that $u_l(xa) = uxa = u_l(x)a$ and hence $u_l \in B$. In particular, since I is a B-module, we may write $zu_l = u_l(z) = uz$ for $u, z \in I$. Now, for $x \in I$, we have that

$$((ay)_I f)(x) = f((ay)_I(x)) = f(xay) = f((xa)y) = f(y(xa)_l)$$

$$= f(y)(xa)_l = xaf(y) = (af(y))_I(x)\,.$$

Since this is for all $x \in I$, we have that

$$(ay)_I f = (af(y))_I \in A_I\,.$$

This completes the proof of the first part of the theorem. The second follows from the first and Schur's lemma. ∎

There is a good reason why the previous theorem is called the double centralizer theorem. With the same notation, notice that, by definition, $\mathrm{End}_A(I) = \mathbf{C}_{\mathrm{End}_F(I)}(A_I)$. Also, $\mathrm{End}_B(I) = \mathbf{C}_{\mathrm{End}_F(I)}(\mathrm{End}_A(I))$. Hence, the double centralizer theorem proves that

$$A_I = \mathbf{C}_{\mathrm{End}_F(I)}(\mathbf{C}_{\mathrm{End}_F(I)}(A_I))\,.$$

An **idempotent** of A (any ring) is a nonzero element $e \in A$ such that $e^2 = e$. Two idempotents e and f are **orthogonal** if $ef = fe = 0$. An idempotent e is **primitive** if it is not the sum of two orthogonal idempotents.

The following key result classifies the semisimple algebras over algebraically closed fields: they are direct sums of matrix algebras.

(1.17) THEOREM (Wedderburn). *Suppose that A is a semisimple algebra over F, an algebraically closed field.*

(a) There is only a finite set $\{B_1, \ldots, B_n\}$ of distinct minimal ideals of A. Also,

$$A = \bigoplus_{j=1}^{n} B_j\,.$$

(b) If I_j is a minimal right ideal of A contained in B_j, then $\{I_1, \ldots, I_n\}$ is a complete set of representatives of pairwise nonisomorphic minimal right ideals of A. In particular, $\{I_1, \ldots, I_n\}$ is a complete set of representatives of pairwise nonisomorphic simple A-modules. Moreover,

$$\mathrm{ann}(I_i) = \sum_{j \neq i} B_j\,.$$

(c) If we write $1 = e_1 + \ldots + e_n$ for $e_i \in B_i$, then every central idempotent of A is a sum of some e_i's. Hence, the complete set of distinct primitive idempotents of $\mathbf{Z}(A)$ is $\{e_1, \ldots, e_n\}$. Also, $B_j = e_jA$ and B_j is a simple F-algebra with identity e_j.

(d) *The natural map induced by right multiplication yields an isomorphism* $B_j \cong \mathrm{End}_F(I_j)$. *In particular,*

$$\dim_F(A) = \sum_{j=1}^{n} (\dim_F(I_j))^2 .$$

(e) *The ideal* B_j *is the direct sum of* $\dim_F(I_j)$ *minimal right ideals of* A *isomorphic to* I_j.

Proof. We prove part (a) by induction on $\dim_F(A)$.

Let B be a minimal ideal of A. Since A is semisimple, we know that the regular A-module is completely reducible by Theorem (1.13). Therefore, there exists a right ideal I such that $A = B \oplus I$. Hence, $IB \subseteq I \cap B = 0$ and thus $I \subseteq X = \{a \in A \mid aB = 0\}$. Since B is an ideal, it is clear that X is an ideal of A. Now, $B^2 \neq 0$ (because $0 = \mathbf{J}(A)$ contains every nilpotent ideal of A) and it follows that B is not contained in X. Since B is a minimal ideal of A, we have that $B \cap X = 0$. However, since $X = I \oplus (B \cap X)$, it follows that $I = X$ is a two sided ideal of A. Also, $IB = BI = 0$ since $BI \subseteq B \cap I = 0$. Notice now that $J \subseteq I$ is an ideal of I if and only if J is an ideal of A (again, because $IB = BI = 0$). Furthermore, if $C \neq B$ is a minimal ideal of A, by minimality we have that $C \cap B = 0$. Thus, $CB = BC \subseteq B \cap C = 0$ and hence $C \subseteq X = I$. We know by Corollary (1.14) that A/B is semisimple. It is clear that the projection map $A \to I$ is a linear map with kernel B, which is onto, multiplicative and mapping the identity of A to what is the identity of I. Hence, I is a semisimple F-algebra and the proof of part (a) clearly follows by induction.

Since $\dim_F(A)$ is finite, we may find a minimal right ideal I_j of A contained in B_j for every j. Now, let J be any minimal right ideal of A. We prove that J is isomorphic to some I_i. Clearly,

$$J = JA = JB_1 + \ldots + JB_n ,$$

and thus there exists an i such that $JB_i \neq 0$. Since JB_i is a right ideal of A, we have that $JB_i = J$ by the minimality of J. In particular, $J \subseteq B_i$ since B_i is an ideal of A. Also, $JB_j = JB_iB_j = 0$ for all $j \neq i$. Now, $\mathrm{ann}(J)$ is an ideal of A which contains B_j for every $j \neq i$. Hence, B_i is not contained in $\mathrm{ann}(J)$ and thus $\mathrm{ann}(J) \cap B_i = 0$ (because B_i is a minimal ideal). Therefore, $JI_i \neq 0$. This implies that there exists an $x \in J$ such that $xI_i \neq 0$. Now, since xI_i is a nonzero right ideal of A contained in the minimal right ideal J, we have that $xI_i = J$. Since the map $y \mapsto xy$ from $I_i \to J$ is a nonzero A-homomorphism between simple modules, we have that $y \mapsto xy$ is an isomorphism by Schur's lemma. This proves that every