

Cambridge University Press
978-0-521-59362-5 - Introduction to Geometric Probability
Daniel A. Klain and Gian-Carlo Rota
Frontmatter
[More information](#)

Introduction to Geometric Probability

This is the first modern introduction to geometric probability, also known as integral geometry. The subject is presented at an elementary level, requiring little more than first year graduate mathematics. The theory of intrinsic volumes due to Hadwiger, McMullen, Santaló and others is presented, along with a complete and elementary proof of Hadwiger's characterization theorem of invariant measures in Euclidean n -space. The theory of the Euler characteristic is developed from an integral-geometric point of view. The authors then prove the fundamental theorem of integral geometry, namely the kinematic formula. Finally the analogies between invariant measures on polyconvex sets and measures on order ideals of finite partially ordered sets are investigated. The relationship between convex geometry and enumerative combinatorics motivates much of the presentation. Every chapter concludes with a list of unsolved problems. Geometers and combinatorialists will find this a stimulating and fruitful tale.

Daniel A. Klain is Assistant Professor of Mathematics at Georgia Institute of Technology.

Gian-Carlo Rota is Professor of Applied Mathematics and Philosophy, Massachusetts Institute of Technology.

Cambridge University Press
978-0-521-59362-5 - Introduction to Geometric Probability
Daniel A. Klain and Gian-Carlo Rota
Frontmatter
[More information](#)

Lezioni Lincee

Sponsored by *Fondazione IBM Italia*

Editor: Luigi A. Radicati di Brozolo, Scuola Normale Superiore, Pisa

This series of books arises from lectures given under the auspices of the Accademia Nazionale dei Lincei and is sponsored by *Fondazione IBM Italia*.

The lectures, given by international authorities, will range on scientific topics from mathematics and physics through to biology and economics. The books are intended for a broad audience of graduate students and faculty members, and are meant to provide a '*mise au point*' for the subject with which they deal. The symbol of the Accademia, the lynx, is noted for its sharp-sightedness; the volumes in this series will be penetrating studies of scientific topics of contemporary interest.

Already published

Chaotic Evolution and Strange Attractors: D. Ruelle

Introduction to Polymer Dynamics: P. de Gennes

The Geometry and Physics of Knots: M. Atiyah

Attractors for Semigroups and Evolution Equations: O. Ladyzhenskaya

Asymptotic Behaviour of Solutions of Evolutionary Equations: M. I. Vishik

Half a Century of Free Radical Chemistry: D. N. R. Barton in collaboration with
S. I. Parekh

Bound Carbohydrates in Nature: L. Warren

Neural Activity and the Growth of the Brain: D. Purves

Perspectives in Astrophysical Cosmology: M. Rees

Molecular Mechanisms in Striated Muscle: S. V. Perry

Some Asymptotic Problems in the Theory of Partial Differential Equations:
O. Oleinik

Cambridge University Press

978-0-521-59362-5 - Introduction to Geometric Probability

Daniel A. Klain and Gian-Carlo Rota

Frontmatter

[More information](#)

Introduction to Geometric Probability

DANIEL A. KLAIN

GIAN-CARLO ROTA



Cambridge University Press
978-0-521-59362-5 - Introduction to Geometric Probability
Daniel A. Klain and Gian-Carlo Rota
Frontmatter
[More information](#)

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore,
São Paulo, Delhi, Dubai, Tokyo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521593625

© Cambridge University Press 1997

This publication is in copyright. Subject to statutory exception
and to the provisions of relevant collective licensing agreements,
no reproduction of any part may take place without the written
permission of Cambridge University Press.

First published 1997
Reprinted 1999

A catalogue record for this publication is available from the British Library

ISBN 978-0-521-59362-5 Hardback
ISBN 978-0-521-59654-1 Paperback

Transferred to digital printing 2010

Cambridge University Press has no responsibility for the persistence or
accuracy of URLs for external or third-party internet websites referred to in
this publication, and does not guarantee that any content on such websites is,
or will remain, accurate or appropriate. Information regarding prices, travel
timetables and other factual information given in this work are correct at
the time of first printing but Cambridge University Press does not guarantee
the accuracy of such information thereafter.

Contents

<i>Preface</i>	<i>page</i> xi
<i>Using this book</i>	xiv
1 The Buffon needle problem	1
1.1 The classical problem	1
1.2 The space of lines	3
1.3 Notes	5
2 Valuation and integral	6
2.1 Valuations	6
2.2 Groemer's integral theorem	8
2.3 Notes	11
3 A discrete lattice	13
3.1 Subsets of a finite set	13
3.2 Valuations on a simplicial complex	21
3.3 A discrete analogue of Helly's theorem	28
3.4 Notes	29
4 The intrinsic volumes for parallelotopes	30
4.1 The lattice of parallelotopes	30
4.2 Invariant valuations on parallelotopes	35
4.3 Notes	41
5 The lattice of polyconvex sets	42
5.1 Polyconvex sets	42
5.2 The Euler characteristic	46
5.3 Helly's theorem	50
5.4 Lutwak's containment theorem	54
5.5 Cauchy's surface area formula	55
5.6 Notes	58

6	Invariant measures on Grassmannians	60
6.1	The lattice of subspaces	60
6.2	Computing the flag coefficients	63
6.3	Properties of the flag coefficients	70
6.4	A continuous analogue of Sperner's theorem	73
6.5	A continuous analogue of Meshalkin's theorem	77
6.6	Helly's theorem for subspaces	81
6.7	Notes	83
7	The intrinsic volumes for polyconvex sets	86
7.1	The affine Grassmannian	86
7.2	The intrinsic volumes and Hadwiger's formula	87
7.3	An Euler relation for the intrinsic volumes	93
7.4	The mean projection formula	94
7.5	Notes	95
8	A characterization theorem for volume	98
8.1	Simple valuations on polyconvex sets	98
8.2	Even and odd valuations	106
8.3	The volume theorem	109
8.4	The normalization of the intrinsic volumes	111
8.5	Lattice points and volume	112
8.6	Remarks on Hilbert's third problem	115
8.7	Notes	117
9	Hadwiger's characterization theorem	118
9.1	A proof of Hadwiger's characterization theorem	118
9.2	The intrinsic volumes of the unit ball	120
9.3	Crofton's formula	123
9.4	The mean projection formula revisited	125
9.5	Mean cross-sectional volume	128
9.6	The Buffon needle problem revisited	129
9.7	Intrinsic volumes on products	130
9.8	Computing the intrinsic volumes	135
9.9	Notes	140
10	Kinematic formulas for polyconvex sets	146
10.1	The principal kinematic formula	146
10.2	Hadwiger's containment theorem	150
10.3	Higher kinematic formulas	152
10.4	Notes	153

<i>Contents</i>		ix
11	Polyconvex sets in the sphere	154
11.1	Convexity in the sphere	154
11.2	A characterization for spherical area	156
11.3	Invariant valuations on spherical polytopes	159
11.4	Spherical kinematic formulas	162
11.5	Remarks on higher dimensional spheres	164
11.6	Notes	166
	<i>Bibliography</i>	168
	<i>Index of symbols</i>	174
	<i>Index</i>	176

Cambridge University Press

978-0-521-59362-5 - Introduction to Geometric Probability

Daniel A. Klain and Gian-Carlo Rota

Frontmatter

[More information](#)

Preface

If we were allowed to rename the field of geometric probability – sometimes already renamed integral geometry – then we would be tempted to choose the oxymoron ‘continuous combinatorics.’ On more than one occasion the two fields, geometric probability and enumerative combinatorics, are brought together by mathematical analogy, that most effective breaker of barriers.

Like combinatorial enumeration, where sequences of objects bearing a common feature are unified by the idea of a generating function, geometric probability studies sets of geometric objects bearing a common feature, which are unified by the idea of an invariant measure. The basic idea is extremely simple. When considering straight lines, pairs of points, or triangles in space, one determines the invariant measure on the variety of straight lines, of pairs of points, of triangles. This idea is strangely reminiscent of the underlying idea of enumerative geometry, with one major difference: whereas enumerative geometry is bound to the counting of finite sets, geometric probability is given greater freedom, by extending the concept of enumeration to allow the assigning of invariant measures. Invariant measures are far easier to compute and, we dare add, more useful than the curiously large integers that are computed in enumerative geometry. This basic idea goes back to Crofton’s article in the ninth edition of the *Encyclopaedia Britannica*, an article that created the subject from scratch and that is still worth reading today. The one other brilliant contribution to geometric probability in the past century was Barbier’s solution of the Buffon needle problem, which remains to this day the basic trick of the subject, still being secretly exploited in ever unsuspected ways.

Geometric probability has suffered in this century the fate of other fields that would have enjoyed a healthy autonomous development, had

it not been for the overpowering development of representation theory. One can reduce integral geometry to the study of actions of Lie groups, to symmetric spaces, to the Radon transform; in so doing, however, the authentic problematic of the subject is lost. Geometric probability is a customer of representation theory, in the same sense that mechanics is a customer of the calculus.

The purpose of this book is to present the three basic ideas of geometric probability, stripped of all reliance on group-theoretic techniques. First, we investigate measures on polyconvex sets (i.e., finite unions of compact convex sets) in Euclidean spaces of arbitrary dimension that are invariant under the group of Euclidean motions. A great many mathematicians are still basking in the illusion that there is only one such measure, namely, the volume. We merrily destroy this illusion by proving what is at present the fundamental result of the field (due to Hadwiger), stating that the space of such invariant measures is of dimension $n + 1$ in a Euclidean space of dimension n . The proof of this fundamental result given in the text is new, due to the first author. It becomes clear, on reading the applications of the fundamental theorem, that the basic invariant measure to be singled out from such a bounty is not the volume, but the Euler characteristic (as Steve Schanuel was first to realize). Here again we meet with wide ignorance on the part of the mathematical public: the fundamental fact that the Euler characteristic is an invariant measure (in fact, it is the only integer-valued invariant measure) is not as well known as it should be. It leads to one-line proofs of most of the fundamental theorems on convex sets. We develop the theory of the Euler characteristic from scratch, in a way that makes it look like an ordinary integral.

Second, we prove the fundamental formula of integral geometry, viz., the kinematic formula. Here we displace the common device of Minkowski sums from its typically central role, not merely as a display of mathematical machismo, but with an ulterior motive.

Third, we try to bring out from the beginning the striking analogy between the computation of invariant measures and certain combinatorial properties of finite partially ordered sets. The second author pointed out in 1967 that the notion of Euler characteristic could be extended to such partially ordered sets by means of the Möbius function. We now go one step further and show that an analogue of the theory of invariant measures in Euclidean space can be worked out in partially ordered sets, including finite analogues of the kinematic formula and even of Helly's theorem. This analogy brings out in stark contrast the

Cambridge University Press

978-0-521-59362-5 - Introduction to Geometric Probability

Daniel A. Klain and Gian-Carlo Rota

Frontmatter

[More information](#)*Preface*

xiii

unexplored terrain of classical geometric probability, namely, a thorough understanding of the integral geometric structure of the lattice of subspaces of Euclidean space under the action of the orthogonal group. It also brings us closer to the current outer limits of mathematics, to the theory of Hecke algebras, to Schubert varieties and to the quantum world.

We hope that the reading of this introduction to the field of geometric probability will encourage further development of these analogies.

The text is based on the ‘Lezioni Lincee’ given by the second author in 1986 at the Scuola Normale Superiore in Pisa. The authors wish to thank Ennio De Giorgi, Edoardo Vesentini, and Luigi Radicati for providing an interested audience for the original lectures. Thanks are also due to Stefano Mortola for his careful reading of the initial draft, and to Beifang Chen, Steve Fisk, Joseph Fu, Steven Holt, Erwin Lutwak, and three anonymous referees for their valuable comments and suggestions.

Using this book

Although parts of this book assume a knowledge of basic point-set topology, measure theory, and elementary probability theory, the greater part of the text should be accessible to advanced undergraduates. Proofs are either given in full or else stretched to the point from which the reader will be able to reconstruct them without effort. Only on certain technical measure-theoretic points have we felt the need to omit details that, although indispensable in a detailed treatment, are of questionable relevance in an exposition that is meant to stress geometric insight and combinatorial analogy. Some notions that appear vague in the early sections will be revisited later on, after language has been developed for a treatment in clear and rigorous terms. References and open problems are deferred to the notes at the end of each chapter.