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131 Bipartite Graphs and their Applications

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Preface

The graphs in figure 0.1.1 all have one property in common: their vertices can be divided into two parts such that no two vertices in the same part are joined by an edge. In the diagrams we have indicated one possible division by colouring the vertices black and white. Graphs which have this property are called bipartite, and their properties form the subject of this book.

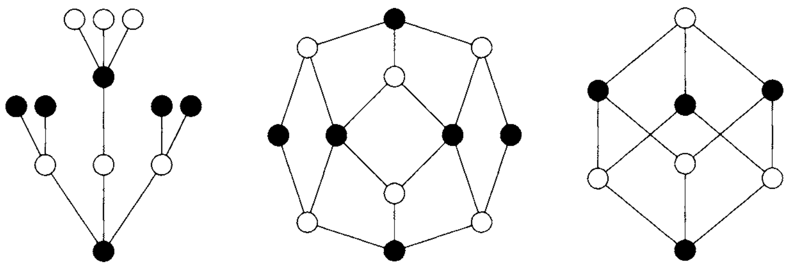

Figure 0.1.1

The first systematic investigation of the properties of bipartite graphs was begun by König (1914, 1915, 1916). His work was motivated by an attempt to give a new approach to the investigation of matrices, in particular to find a simple proof of a theorem of Frobenius (1912) on determinants of matrices. Much of the material from this early work is presented in König's famous book (1936). But this is not the beginning of the story. Trees, a large subclass of bipartite graphs, had already been defined and investigated by several authors much earlier. Kirchhoff (1847), Cayley (1857), Sylvester (1873) and Jordan (1869) each independently developed theories of trees in the mid nineteenth century.

From a practical point of view, bipartite graphs form a model of the interaction between two different types of objects, be they sets and their elements, jobs and workers, or telephone exchanges and cities. The desire to model such interaction is extremely common, and many recreational and much more serious problems can be phrased in terms of problems on bipartite graphs.

From a theoretical point of view, bipartite graphs at first glance seem to have a much simpler structure than graphs in general, but this is not altogether the case. They are certainly much easier to envisage, but actually almost all the difficulties which are inherent in general graphs are already present in bipartite graphs. For instance, wide classes of computational problems are already NP-hard even for bipartite graphs with small maximal degrees. Historically, many results on bipartite graphs have been the starting points for various generalisations to results on general graphs. For these reasons alone, bipartite graphs have been considered in every book about graph theory, but up until now only as a special class in some wider context.

However, over the past few years many new and interesting results about bipartite graphs have been obtained, which are difficult to include in a traditional book on graph theory. For this reason, among others, we decided to write this, the first book about bipartite graphs alone. Our aim was to describe properties of this class of graphs using only their own structure, together with occasionally some simple algebraic techniques. We have by and large avoided results which are proved using random methods and linear programming (the interested reader can find such material in the books of Alon and Spencer (1992), Bollobás (1985), Ford and Fulkerson (1962) and Lawler (1976)).

Together with the traditional topics which we have tried to present in a new light, the reader will also find many new and unusual results. In particular, we have concentrated much attention on edge colouring problems with a variety of restrictions, and their applications to timetabling. We also give applications of many other results in the areas of algebra, combinatorics, chemistry, communication networks and computer science.

Let us briefly survey the contents of the chapters, pointing out some of the highlights. We begin from the beginning, and introduce the basic tools and concepts to be used throughout the book. Next follow the basic characterisation theorems of both general bipartite graphs and some particular special classes. We move on to some results about global structure, first considering metric properties of bipartite graphs, trees and hypercubes in particular, and then results concerning connectivity. A method for efficiently routing information about a computer network, and a construction of linear superconcentrators which are sparse and yet highly connected networks, form the applications for these chapters.

Next is a chapter about various types of matching problems. We begin with maximum matchings, both with their properties and with the practical task of actually finding such an object. Together with the well-known algorithm of Hopcroft and Karp (1973) to find a maximum matching, we also give a modified algorithm due to Alt, Blum, Mehlhorn and Paul (1991). One section is devoted to stable matchings, and the last section to a polynomial algorithm for finding the k best perfect matchings in a weighted complete bipartite graph.

The chapter which follows begins from somewhat similar ground, but from a slightly different standpoint. It is concerned with what can be said about graphs which have a so-called 'expanding' property, with some additive or multiplicative factor. The chapter continues to the more contemporary subject of expanders, and we show how the structure of these graphs can be used for powerful sorting algorithms.

Chapter 7 is concerned with subgraphs which satisfy degree restrictions of one sort or another. We discuss existence theorems for (g, f)-factors, and f-factors, and go on to discuss the important special cases of 2-factors, and connected 2-factors (or, as they are better known, Hamilton cycles). Restricting only the parity of the vertex degrees, T-joins are also considered.

Of all bipartite graph topics, edge colourings are among the most widely studied problems in the field. In the chapter on edge colourings, there are a variety of results about the existence of edge colourings under many different restrictions and with many different properties. Edge colourings of bipartite graphs have a rather natural interpretation in terms of timetables. We explore this interplay quite fully throughout the chapter; indeed many of the styles of restrictions we consider are inspired by timetabling scenarios. Among the results is a proof of the now famous conjecture about the list colouring problem for bipartite graphs.

The connection between bipartite graphs and matrices with non-negative entries is fundamental both historically and in reality. In this chapter we give the theorem of Birkhoff and von Neumann on a decomposition of a doubly stochastic matrix, and a criterion for when this decomposition is unique. As an application, but this time in somewhat the opposite sense to elsewhere, we apply a proof of van der Waerden's conjecture, to give results about numbers of perfect matchings in bipartite graphs.

The chapter on coverings begins with a formulation of a general covering problem. Throughout this chapter we give a broad spectrum of different instances of this problem in the context of bipartite graphs.

The penultimate chapter employs techniques and results from throughout the book, to prove a collection of combinatorial results. Theorems of edge colouring, connectivity and Hamilton cycles are used to prove results about Gray codes, systems of distinct representatives, sums of integer divisors, and completing partial latin squares.

In the final chapter we turn to problems concerning bipartite subgraphs of arbitrary graphs. We give results on the maximum size of bipartite subgraphs and several forms of edge coverings and edge-decompositions by bipartite graphs. The book ends with an appendix of NP-complete problems concerning bipartite graphs.

This book is based on graduate courses taught by A.S. Asratian at Yerevan State University and lectures given by R. Häggkvist at Umeå University. The selection of the material was of course heavily influenced by our personal interests and also limitations of space. For the most part the material discussed is accessible to any reader with a graduate understanding of mathematics. However, the book also contains advanced sections requiring much more specialised knowledge, which will be of interest to specialists in combinatorics and graph theory.

Although we include exercises designed to clarify the material, many of the exercises given are present more to supplement the results in the text. We have made some attempt to indicate the difficulty of each exercise. The easiest exercises are denoted with a $\triangle$, exercises which are hard or require much work are denoted with a ∇, and a few very challenging exercises which require techniques beyond this book are denoted with a $\square$. The remaining exercises are deemed to be tractable. The book would serve amply for teaching graduate courses, as well as a reference text.

We send thanks to our colleagues at the universities in Cambridge, Moscow, Twente, Umeå and Yerevan, for their help and remarks. We are also thankful to C. Berge, B. Bollobás, J.A. Bondy, N.N. Kuzjurin and O.B. Lupanov for their encouragement at various stages of preparation. Dr. Asratian would also like to thank the Netherlands Organisation for Scientific Research and the Natural Sciences and Engineering Research Council of Canada for partial support. Finally, and most wholeheartedly, we would like to thank the University of Umeå for providing the opportunity for this venture.

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Notation

$V(G)$	the vertex set of a graph G
$E(G)$	the edge set of G
$d_G(v)$	the degree of a vertex v in G
$\delta(G)$	the minimum degree of G
$\Delta(G)$	the maximum degree of G
$G[W]$	the subgraph of G induced by the vertex or edge set W
$d_G(u,v)$	the distance between vertices u and v in G
$N_i(v)$	the set of vertices at distance i from v
$N(S)$	the set of vertices which are neighbours of some vertex in S
$G-S$	the subgraph of G obtained by deleting the set of vertices S
$G-v$	the subgraph of G obtained by deleting the vertex v
$G-e$	the subgraph of G obtained by deleting the edge e
$G+e$	the graph obtained by adding the edge e to G
$E_G(X,Y)$	the set of edges joining a vertex in X to a vertex in Y in G
$\mathbf{M}(G)$	the incidence matrix of G
$\mathbf{A}(G)$	the adjacency matrix of G
$\mathbf{B}(G)$	the biadjacency matrix of G
$\chi'(G)$	the chromatic index of G
$r(G)$	the radius of G
$d(G)$	the diameter of G
$\kappa(G)$	the vertex connectivity of G
$\lambda(G)$	the edge connectivity of G
$bip(G)$	the graph obtained by subdividing every edge of G
(V_1,V_2)	a bipartition of a bipartite graph
$H \subseteq G$	the graph H is a subgraph of the graph G
$H \subset G$	the graph H is a proper subgraph of the graph G
$G \times H$	the graph product of the graphs G and H
uv	an edge joining vertices u and v in a simple graph
$K_{r,s}$	complete bipartite graph with colour classes of r & s vertices
K_n	the complete graph on n vertices
Q_n	the n–dimensional hypercube
$M(f,r)$	the set of edges coloured r in the edge colouring f
$\rho(\boldsymbol{\alpha},\boldsymbol{\beta})$	the Hamming distance between $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$
$\vec{G}$	an orientation of the graph G
$\overrightarrow{uv}$	a directed edge from u to v in a simple directed graph
$\lceil x \rceil$	the smallest integer at least as large as x
$\lfloor x \rfloor$	the greatest integer at most as large as x
S_n	the set of permutations of the set $\{1,\ldots,n\}$
$A \triangle B$	the symmetric difference of the sets A and B
$A \subseteq B$	A is a subset of B
$A \subset B$	A is a proper subset of B