QUANTUM FIELD THEORY AND
CONDENSED MATTER

Providing a broad review of many techniques and their application to condensed matter systems, this book begins with a review of thermodynamics and statistical mechanics, before moving on to real- and imaginary-time path integrals and the link between Euclidean quantum mechanics and statistical mechanics. A detailed study of the Ising, gauge-Ising and XY models is included. The renormalization group is developed and applied to critical phenomena, Fermi liquid theory, and the renormalization of field theories. Next, the book explores bosonization and its applications to one-dimensional fermionic systems and the correlation functions of homogeneous and random-bond Ising models. It concludes with the Bohm–Pines and Chern–Simons theories applied to the quantum Hall effect. Introducing the reader to a variety of techniques, it opens up vast areas of condensed matter theory for both graduate students and researchers in theoretical, statistical, and condensed matter physics.

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QUANTUM FIELD THEORY AND
CONDENSED MATTER

An Introduction

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Dedicated to
Michael Fisher, Leo Kadanoff, Ben Widom, and Ken Wilson
Architects of the modern RG
## Contents

**Preface**

1. Thermodynamics and Statistical Mechanics Review  
   1.1 Energy and Entropy in Thermodynamics  
   1.2 Equilibrium as Maximum of Entropy  
   1.3 Free Energy in Thermodynamics  
   1.4 Equilibrium as Minimum of Free Energy  
   1.5 The Microcanonical Distribution  
   1.6 Gibbs’s Approach: The Canonical Distribution  
   1.7 More on the Free Energy in Statistical Mechanics  
   References and Further Reading

2. The Ising Model in $d = 0$ and $d = 1$  
   2.1 The Ising Model in $d = 0$  
   2.2 The Ising Model in $d = 1$  
   2.3 The Monte Carlo Method

3. Statistical to Quantum Mechanics  
   3.1 Real-Time Quantum Mechanics  
   3.2 Imaginary-Time Quantum Mechanics  
   3.3 The Transfer Matrix  
   3.4 Classical to Quantum Mapping: The Dictionary  
   References and Further Reading

4. Quantum to Statistical Mechanics  
   4.1 From $U$ to $Z$  
   4.2 A Detailed Example from Spin $\frac{1}{2}$  
   4.3 The $\tau$-Continuum Limit of Fradkin and Susskind  
   4.4 Two $N \to \infty$ Limits and Two Temperatures  
   References and Further Reading
Contents

5 The Feynman Path Integral 52
  5.1 The Feynman Path Integral in Real Time 52
  5.2 The Feynman Phase Space Path Integral 58
  5.3 The Feynman Path Integral for Imaginary Time 59
  5.4 Classical ↔ Quantum Connection Redux 60
  5.5 Tunneling by Euclidean Path Integrals 61
  5.6 Spontaneous Symmetry Breaking 66
  5.7 The Classical Limit of Quantum Statistical Mechanics 70
References and Further Reading 71

6 Coherent State Path Integrals for Spins, Bosons, and Fermions 72
  6.1 Spin Coherent State Path Integral 72
  6.2 Real-Time Path Integral for Spin 74
  6.3 Bosonic Coherent States 76
  6.4 The Fermion Problem 78
  6.5 Fermionic Oscillator: Spectrum and Thermodynamics 78
  6.6 Coherent States for Fermions 80
  6.7 Integration over Grassmann Numbers 82
  6.8 Resolution of the Identity and Trace 85
  6.9 Thermodynamics of a Fermi Oscillator 86
  6.10 Fermionic Path Integral 86
  6.11 Generating Functions $Z(J)$ and $W(J)$ 94
References and Further Reading 104

7 The Two-Dimensional Ising Model 105
  7.1 Ode to the Model 105
  7.2 High-Temperature Expansion 107
  7.3 Low-Temperature Expansion 108
  7.4 Kramer–Wannier Duality 110
  7.5 Correlation Function in the tanh Expansion 112
References and Further Reading 113

8 Exact Solution of the Two-Dimensional Ising Model 114
  8.1 The Transfer Matrix in Terms of Pauli Matrices 114
  8.2 The Jordan–Wigner Transformation and Majorana Fermions 115
  8.3 Boundary Conditions 118
  8.4 Solution by Fourier Transform 120
  8.5 Qualitative Analysis in the $\tau$-Continuum Limit 123
  8.6 The Eigenvalue Problem of $T$ in the $\tau$-Continuum Limit 125
  8.7 Free Energy in the Thermodynamic Limit 134
  8.8 Lattice Gas Model 135
  8.9 Critical Properties of the Ising Model 136
  8.10 Duality in Operator Language 140
References and Further Reading 142
# Contents

9 Majorana Fermions 143
  9.1 Continuum Theory of the Majorana Fermion 143
  9.2 Path Integrals for Majorana Fermions 147
  9.3 Evaluation of Majorana Grassmann Integrals 149
  9.4 Path Integral for the Continuum Majorana Theory 152
  9.5 The Pfaffian in Superconductivity 155
References and Further Reading 156

10 Gauge Theories 157
  10.1 The $\text{XY}$ Model 157
  10.2 The $\mathbb{Z}_2$ Gauge Theory in $d = 2$ 163
  10.3 The $\mathbb{Z}_2$ Theory in $d = 3$ 169
  10.4 Matter Fields 173
  10.5 Fradkin–Shenker Analysis 176
References and Further Reading 182

11 The Renormalization Group 183
  11.1 The Renormalization Group: First Pass 183
  11.2 Renormalization Group by Decimation 186
  11.3 Stable and Unstable Fixed Points 192
  11.4 A Review of Wilson’s Strategy 194
References and Further Reading 198

12 Critical Phenomena: The Puzzle and Resolution 199
  12.1 Landau Theory 201
  12.2 Widom Scaling 207
  12.3 Kadanoff’s Block Spins 210
  12.4 Wilson’s RG Program 214
  12.5 The $\beta$-Function 220
References and Further Reading 223

13 Renormalization Group for the $\phi^4$ Model 224
  13.1 Gaussian Fixed Point 224
  13.2 Gaussian Model Exponents for $d > 4$, $\varepsilon = 4 - d = -|\varepsilon|$ 237
  13.3 Wilson–Fisher Fixed Point $d < 4$ 242
  13.4 Renormalization Group at $d = 4$ 248
References and Further Reading 250

14 Two Views of Renormalization 251
  14.1 Review of RG in Critical Phenomena 251
  14.2 The Problem of Quantum Field Theory 252
  14.3 Perturbation Series in $\lambda_0$: Mass Divergence 253
  14.4 Scattering Amplitude and the $\Gamma$’s 254
  14.5 Perturbative Renormalization 259
  14.6 Wavefunction Renormalization 260
x

## Contents

14.7 Wilson’s Approach to Renormalizing QFT | 264
14.8 Theory with Two Parameters | 271
14.9 The Callan–Symanzik Equation | 273
References and Further Reading | 283

15 Renormalization Group for Non-Relativistic Fermions: I | 285
15.1 A Fermion Problem in $d = 1$ | 289
15.2 Mean-Field Theory $d = 1$ | 291
15.3 The RG Approach for $d = 1$ Spinless Fermions | 294
References and Further Reading | 303

16 Renormalization Group for Non-Relativistic Fermions: II | 305
16.1 Fermions in $d = 2$ | 305
16.2 Tree-Level Analysis of $u$ | 309
16.3 One-Loop Analysis of $u$ | 311
16.4 Variations and Extensions | 315
References and Further Reading | 317

17 Bosonization I: The Fermion–Boson Dictionary | 319
17.1 Preamble | 319
17.2 Massless Dirac Fermion | 321
17.3 Free Massless Scalar Field | 324
17.4 Bosonization Dictionary | 328
17.5 Relativistic Bosonization for the Lagrangians | 332
References and Further Reading | 333

18 Bosonization II: Selected Applications | 334
18.1 Massless Schwinger and Thirring Models | 334
18.2 Ising Correlations at Criticality | 336
18.3 Random-Bond Ising Model | 340
18.4 Non-Relativistic Lattice Fermions in $d = 1$ | 346
18.5 Kosterlitz–Thouless Flow | 357
18.6 Analysis of the KT Flow Diagram | 360
18.7 The XXZ Spin Chain | 363
18.8 Hubbard Model | 364
18.9 Conclusions | 367
References and Further Reading | 367

19 Duality and Triality | 370
19.1 Duality in the $d = 2$ Ising Model | 370
19.2 Thirring Model and Sine-Gordon Duality | 373
19.3 Self-Triality of the $SO(8)$ Gross–Neveu Model | 376
19.4 A Bonus Result: The $SO(4)$ Theory | 382
References and Further Reading | 383
Preface

Condensed matter theory is a massive field to which no book or books can do full justice. Every chapter in this book is possible material for a book or books. So it is clearly neither my intention nor within my capabilities to give an overview of the entire subject. Instead I will focus on certain techniques that have served me well over the years and whose strengths and limitations I am familiar with.

My presentation is at a level of rigor I am accustomed to and at ease with. In any topic, say the renormalization group (RG) or bosonization, there are treatments that are more rigorous. How I deal with this depends on the topic. For example, in the RG I usually stop at one loop, which suffices to make the point, with exceptions like wave function renormalization where you need a minimum of two loops. For non-relativistic fermions I am not aware of anything new one gets by going to higher loops. I do not see much point in a scheme that is exact to all orders (just like the original problem) if in practice no real gain is made after one loop. In the case of bosonization I work in infinite volume from the beginning and pay scant attention to the behavior at infinity. I show many examples where this is adequate, but point to cases where it is not and suggest references. In any event I think the student should get acquainted with these more rigorous treatments after getting the hang of it from the treatment in this book. I make one exception in the case of the two-dimensional Ising model where I pay considerable attention to boundary conditions, without which one cannot properly understand how symmetry breaking occurs only in the thermodynamic limit.

This book has been a few years in the writing and as a result some of the topics may seem old-fashioned; on the other hand, they have stood the test of time.

Ideally the chapters should be read in sequence, but if that is not possible, the reader may have to go back to earlier chapters when encountering an unfamiliar notion.

I am grateful to the Aspen Center for Physics (funded by NSF Grant 1066293) and the Indian Institute of Technology, Madras for providing the facilities to write parts of this book.

Over the years I have drawn freely on the wisdom of my collaborators and friends in acquiring the techniques described here. I am particularly grateful to my long-standing collaborator Ganpathy Murthy for countless discussions over the years.

xiii
Preface

Of all the topics covered here, my favorite is the renormalization group. I have had the privilege of interacting with its founders: Michael Fisher, Leo Kadanoff, Ben Widom, and Ken Wilson. I gratefully acknowledge the pleasure their work has given me while learning it, using it, and teaching it.

In addition, Michael Fisher has been a long-time friend and role model – from his exemplary citizenship and his quest for accuracy in thought and speech, right down to the imperative to punctuate all equations. I will always remain grateful for his role in ensuring my safe passage from particle theory to condensed-matter theory.

This is also a good occasion to acknowledge the 40+ happy years I have spent at Yale, where one can still sense the legacy of its twin giants: Josiah Willard Gibbs and Lars Onsager. Their lives and work inspire all who become aware of them.

To the team at Cambridge University Press, my hearty thanks: Editor Simon Capelin for his endless patience with me and faith in this book, Roisin Munnelly and Helen Flitton for document handling, and my peerless manuscript editor Richard Hutchinson for a most careful reading of the manuscript and correction of errors in style, syntax, punctuation, referencing, and, occasionally, even equations! Three cheers for university presses, which exemplify what textbook publication is all about.

Finally I thank the three generations of my family for their love and support.