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Mathematical preliminaries

For nothing is that errs from law
In Memoriam A.H.H. LXXIII

Science moves, but slowly slowly, creeping on from point to point
Locksley Hall

Before we commence our presentation of the theory of water waves, we require a firm and precise base from which to start. This must be, at the very least, a statement of the relevant governing equations and boundary conditions. However, it is more satisfactory, we believe, to provide some background to these equations, albeit within the confines of an introductory and relatively brief chapter. The intention is therefore to present a derivation of the equations for inviscid fluid mechanics (*Euler's equation* and the equation of *mass conservation*) and a few of their properties. (The corresponding equations for a viscous fluid – primarily the *Navier–Stokes equation* – appear in Appendix A.) Coupled to these general equations is the set of boundary (and initial) conditions which select the water-wave problem from all other possible solutions of the equations. Of particular importance, as we shall see, are the conditions that define and describe the surface of the fluid; these include the *kinematic condition* and the rôles of *pressure* and *surface tension*. Some rather general consequences of coupling the equations and boundary conditions will also be mentioned.

Once we have available the complete prescription of the water-wave problem, based on a particular model (such as for inviscid flow), we may 'normalise' in any manner that is appropriate. It turns out to be very convenient – and is indeed typical of the applied mathematical approach – to introduce a suitable set of *nondimensional variables*. Further, a useful next step (which is particularly significant for our work in Chapters 3 and 4) is to *scale* the variables with respect to the small parameters thrown up by the nondimensionalisation. All this will enable us to characterise, in a rather precise way, the various types of approximation that we shall employ. In the process, we shall give a summary of the equations that represent different approximations of the full water-wave problem.

Throughout, we take the opportunity to present all the relevant equations in both rectangular Cartesian and cylindrical coordinates.

In the final stage of this preliminary discussion we provide a brief overview of some of the ideas that will permeate many of the problems that we shall encounter. This involves a simple introduction to the mathematics of wave propagation, where we describe the important phenomena associated with the *nonlinearity*, *dispersion* and *dissipation* of the wave. Further, much of our work in the newer aspects of water-wave theory will be with small-amplitude waves and with the slow evolution of wave properties; these may occur separately or together. In order to extract useful and relevant solutions in these cases, we shall require the application of asymptotic methods. Here we present an introduction to the use of *asymptotic expansions*, which will include both near-field and far-field asymptotics and the method of multiple scales.

These mathematical preliminaries may cover material already familiar to some readers, in whole or in part. Those with a background in fluid mechanics could ignore Section 1.1, whereas, for example, those who have received a basic course in wave propagation and elementary asymptotics could ignore Section 1.4. In Chapter 2, and thereafter, we start by giving a summary of the equations and boundary conditions that are relevant to each topic under discussion; this, at its simplest level, is all that is necessary to begin those studies.

1.1 The governing equations of fluid mechanics

In these derivations we shall use a vector notation and the methods of the vector calculus. (The tensor calculus is used in the brief derivation of the Navier–Stokes equation given in Appendix A, although the resulting equation is also written there in terms of vectors.) Here we shall derive the equations of mass conservation and motion (Newton's Second Law) in the absence of thermal changes (which are altogether irrelevant in the propagation of water waves). Any energy equation is therefore a consequence of only the motion (through Newton's Second Law) without any contributions from the thermodynamics of the fluid.

The notation that we shall adopt is the conventional one: at any point in the fluid, the velocity of the fluid is $\mathbf{u}(\mathbf{x}, t)$ where $\mathbf{x}$ is the position vector and t is a time coordinate. The density (mass/unit volume) of the fluid is $\rho(\mathbf{x}, t)$ (but for water-wave applications, as we shall mention later, we take $\rho =$ constant); the pressure at any point in the fluid is $P(\mathbf{x}, t)$. If the choice of coordinates is the familiar right-handed rectangular Cartesian system, then we write

$$\mathbf{x} \equiv (x, y, z) \quad \text{and} \quad \mathbf{u} \equiv (u, v, w).$$

We shall assume that $\mathbf{u}$, ρ, and P are continuous functions (in $\mathbf{x}$ and t) – usually called the *continuum hypothesis* – and that they are also suitably differentiable functions.

1.1.1 The equation of mass conservation

Imagine a volume V, which is bounded by the surface S, within (and totally occupied by) the fluid. We treat V as fixed relative to some chosen inertial frame, so that the fluid in motion may cross the imaginary surface S. Given that the density of the fluid is $\rho(\mathbf{x}, t)$, then the rate of change of mass in V is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathrm{V}} \rho \, \mathrm{d}v\right)$$

where $\int_{\mathrm{V}} \mathrm{d}v$ represents the triple integral over V. Now, let $\mathbf{n}$ be the outward unit normal on S (see Figure 1.1) so that the outward velocity component of the fluid across S is $\mathbf{u} \cdot \mathbf{n}$. Thus the net rate at which mass flows *out* of V is

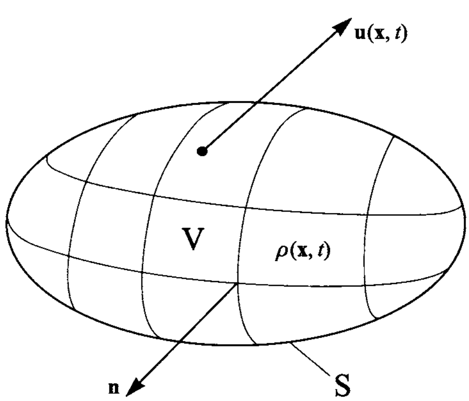

Figure 1.1. The volume V bounded by the surface S; $\rho(x, t)$ is the density of the fluid, $\mathbf{u}(\mathbf{x}, t)$ is the velocity at a point in the fluid and $\mathbf{n}$ is the outward normal on S.

$$\int_{S} \rho \mathbf{u} \cdot \mathbf{n} \mathrm{d}s,$$

where this is the double integral over S.

Under the fundamental assumption that matter (mass) is neither created nor destroyed anywhere in the fluid, the rate of change of mass in V is brought about only by the rate of mass flowing *into* V across S, so

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{V} \rho \,\mathrm{d}v\right) = -\int_{S} \rho \mathbf{u} \cdot \mathbf{n} \mathrm{d}s.$$

This equation is rewritten by the application of *Gauss' theorem* (the *divergence theorem*) to the integral on the right, to give

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{V} \rho \,\mathrm{d}v\right) + \int_{V} \nabla \cdot (\rho \mathbf{u}) \mathrm{d}v = 0$$

where ∇ is the familiar *del* operator (used here in the *divergence* of $\rho \mathbf{u}$). Further, since V is fixed in our coordinate system, the only dependence on t is through $\rho(\mathbf{x}, t)$, so we may write

$$\int_{V} \left\{ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right\} \mathrm{d}v = 0. \tag{1.1}$$

(We shall write more about *differentiation under the integral sign* later; see also Q1.30, Q1.31.) Now equation (1.1) is clearly applicable to any V totally occupied by the fluid, so the limits (represented symbolically by V) of the triple integral are therefore arbitrary; the integral is then always zero (for a continuous integrand, which we assume here) only if

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0. \tag{1.2}$$

This equation, (1.2), is one form of the *equation of mass conservation* (sometimes called the *continuity* equation, referring to the continuity of matter). (The argument that takes us from (1.1) to (1.2) can be rehearsed in the simple example

$$\int_{a}^{b} f(x) \mathrm{d}x = 0 \quad \text{for arbitrary } a, b \Rightarrow f(x) = 0;$$

this is left as an exercise.)

It is usual to expand (1.2) as

$$\frac{\partial \rho}{\partial t} + \rho(\nabla \cdot \mathbf{u}) + (\mathbf{u} \cdot \nabla)\rho = 0,$$

and then introduce

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla, \tag{1.3}$$

the *material* (or *convective*) *derivative*; see Q1.5 and Section 1.2.1. Equation (1.2) therefore becomes

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \nabla \cdot \mathbf{u} = 0, \tag{1.4}$$

from which we see that for an incompressible flow defined by

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0, \tag{1.5}$$

we have

$$\nabla \cdot \mathbf{u} = 0. \tag{1.6}$$

(A function (**u**) which satisfies equation (1.6), so that the divergence of **u** is zero, is said to be *solenoidal*.) Equation (1.5) describes the constancy of ρ on individual fluid particles; we shall, however, interpret incompressibility as meaning $\rho =$ constant everywhere (which is clearly a solution of (1.5), and a very good model for fluids like water). Some of these basic ideas are explored in Q1.7–Q1.9.

1.1.2 The equation of motion: Euler's equation

We now turn our attention to the application of Newton's Second Law to a fluid, but a fluid which is assumed to be *inviscid*; that is, it has zero viscosity (internal friction). (The corresponding equation for a viscous fluid – the Navier–Stokes equation – is described in Appendix A.) Newton's Second Law requires us to balance the rate of change of (linear) momentum of the fluid against the resultant force acting on the fluid. First, therefore, we must find a representation of the forces acting on the fluid.

There are two types of force that are relevant in fluid mechanics: a *body force*, which is more or less the same for all particles and has its source exterior to the fluid, and a *local* (or *short-range*) *force*, which is the force exerted on a fluid element by other elements nearby. The body force which is almost always present is gravity, and this is certainly the case in the study of water waves. We define the general body force to be $\mathbf{F}(\mathbf{x}, t)$ per unit mass; if $\mathbf{F}$ is due solely to the (constant) acceleration of gravity (g) then we would write $\mathbf{F} \equiv (0, 0, -g)$ in both Cartesian and cylindrical coordinates (with z measured positive upwards). The local force is comprised of a pressure contribution together with any viscous forces that are present; in general, of course, this is conveniently represented by the stress tensor in the fluid: see Appendix A. Here we retain only the pressure (P), which produces a *normal* force acting *onto* any element of fluid.

To proceed we define (just as before) an imaginary volume V, bounded by the surface S, which is fixed in our frame of reference and totally occupied by the fluid. The total force (body + local) acting *on* the fluid in V is

$$\int_{V} \rho \mathbf{F} \mathrm{d}v - \int_{S} P\mathbf{n} \mathrm{d}s;$$

see Figure 1.2. (We remember that $\mathbf{n}$ is the *outward* unit normal on S.) Applying Gauss' theorem to the second integral (see Q1.2), we obtain the resultant force

$$\int_{V} (\rho \mathbf{F} - \nabla P) \mathrm{d}v. \qquad (1.7)$$

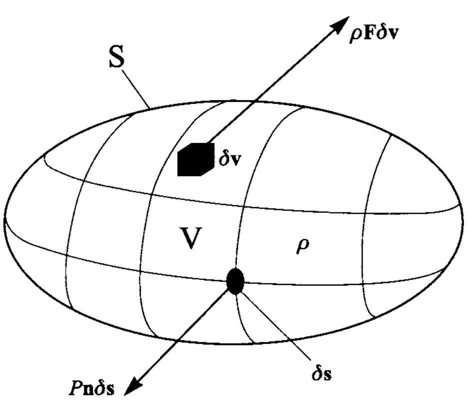

Figure 1.2. The volume V bounded by the surface S; the body force on an element is $\rho\mathbf{F}\delta v$ and the pressure force *on* an element of area is $-P\mathbf{n}\delta s$.

The rate of change of momentum of the fluid in V is simply

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathrm{V}} \rho\,\mathbf{u}\,\mathrm{d}v\right), \tag{1.8}$$

and the rate of flow of momentum across S into V is

$$-\int_{\mathrm{S}} \rho\,\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\,\mathrm{d}s. \tag{1.9}$$

Now Newton's Second Law for the fluid in V (upon recalling that V is fixed in our coordinate frame) may be expressed as:

rate of change of momentum of fluid in V
= resultant force acting on fluid in V
+ rate of flow of momentum across S into V.

Thus from equations (1.7)–(1.9) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathrm{V}} \rho\,\mathbf{u}\,\mathrm{d}v\right) = \int_{\mathrm{V}} (\rho\mathbf{F} - \nabla P)\,\mathrm{d}v - \int_{\mathrm{S}} \rho\,\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\,\mathrm{d}s,$$

which is written more compactly by (a) taking $\mathrm{d}/\mathrm{d}t$ through the integral sign, (b) applying Gauss' theorem to each component of (1.9) (see Q1.3), and, (c), rearranging, to yield

$$\int_{\mathrm{V}} \left\{\frac{\partial}{\partial t}(\rho\,\mathbf{u}) + \rho\,\mathbf{u}(\nabla\cdot\mathbf{u}) + (\mathbf{u}\cdot\nabla)\rho\mathbf{u}\right\}\mathrm{d}v = \int_{\mathrm{V}} (\rho\,\mathbf{F} - \nabla P)\,\mathrm{d}v. \tag{1.10}$$

We expand the integrand on the left side of this equation as

$$\int_{\mathrm{V}} \left\{\rho\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\frac{\partial\rho}{\partial t} + \rho\,\mathbf{u}(\nabla\cdot\mathbf{u}) + \mathbf{u}(\mathbf{u}\cdot\nabla)\rho + \rho(\mathbf{u}\cdot\nabla)\mathbf{u}\right\}\mathrm{d}v = \int_{\mathrm{V}} \rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}\,\mathrm{d}v, \tag{1.11}$$

where we have used the equation of mass conservation, (1.4), and introduced the material derivative, (1.3). It is clear that, with sufficient understanding of the notion of the material derivative (see Q1.4–Q1.6), we could write (1.11) directly: it is the appropriate form of 'mass × acceleration' for all the fluid in V.

The equation (1.10), with (1.11), now becomes

$$\int_V \left(\rho\frac{D\mathbf{u}}{dt} - \rho\mathbf{F} + \nabla P\right)dv = \mathbf{0}$$

and, as before, for this to be valid for arbitrary V (and a continuous integrand) we must have

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla P + \mathbf{F}, \tag{1.12}$$

when written in its usual form. This is *Euler's equation*, which is the result of applying Newton's Second Law to an inviscid (that is, frictionless) fluid. (Notice that the pressure, P, may be defined relative to an arbitrary constant value without altering equation (1.12).)

It is convenient, particularly in view of our later work, to present the three components of Euler's equation, (1.12), and also the equation of mass conservation, in the two coordinate systems that we shall use. In rectangular Cartesian coordinates, $\mathbf{x} \equiv (x, y, z)$, with $\mathbf{u} \equiv (u, v, w)$ and $\mathbf{F} \equiv (0, 0, -g)$, and for constant density, equations (1.12) and (1.6) become, respectively,

$$\left.\begin{aligned} &\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial x}, \quad \frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial y}, \quad \frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial z} - g \\ &\text{where} \\ &\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}, \end{aligned}\right\} \tag{1.13}$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{1.14}$$

These same equations written in cylindrical coordinates, $\mathbf{x} \equiv (r, \theta, z)$, with $\mathbf{u} \equiv (u, v, w)$ (where the same notation for $\mathbf{u}$ in this system should not cause any confusion: it will be plain which coordinates are being used in a given calculation) are, again with $\mathbf{F} \equiv (0, 0, -g)$ and $\rho =$ constant,

$$\left.\begin{aligned} &\frac{Du}{Dt} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r}, \quad \frac{Dv}{Dt} + \frac{uv}{r} = -\frac{1}{\rho}\frac{1}{r}\frac{\partial P}{\partial \theta}, \\ &\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial z} - g \end{aligned}\right\} \tag{1.15}$$

where

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \theta} + w\frac{\partial}{\partial z},$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0. \tag{1.16}$$

These equations, (1.13–1.16), will form the basis for the developments described in Chapters 2, 3, and 4, when coupled to the appropriate boundary conditions (Section 1.2) and – usually – after suitable simplification (Section 1.3). (The corresponding equations for a viscous fluid are presented in Appendix A, and are used in Chapter 5.)

1.1.3 Vorticity, streamlines and irrotational flow

A fundamental property of a fluid flow is the *curl* of the velocity field: $\nabla_{\wedge}\mathbf{u}$. This is called the *vorticity*, and it is conventionally represented by the vector $\boldsymbol{\omega}$; the vorticity measures the local spin or rotation of the fluid (that is, the rotational motion – as compared with the translational) of a fluid element (see Q1.12). In consequence, flows, or regions of flows, in which $\boldsymbol{\omega} \equiv \mathbf{0}$ are said to be *irrotational*; such flows can often be analysed by using particularly routine methods. Unfortunately, real flows are very rarely irrotational anywhere, but for many flows the vorticity is very small almost everywhere, and these may therefore be modelled by assuming irrotationality. Nevertheless, many important aspects of fluid flow require $\boldsymbol{\omega} \neq \mathbf{0}$ somewhere, and the study of such flows normally involves a consideration of the dynamics of vorticity and its properties. In water-wave problems, however, classical aspects of vorticity play a rather minor rôle, and so a deep knowledge of vorticity is not a prerequisite for a study of water waves. (Some small exploration of vorticity is offered in the exercises: see Q1.13–Q1.17.)

Now, before we make use of the vorticity vector in Euler's equation, we introduce a very powerful – but related – concept in the study of fluid motion: the *streamline*. Consider the family of (imaginary) curves which everywhere have the velocity vector as their tangent; these curves are the streamlines. If such a curve is described by $\mathbf{x} = \mathbf{x}(s;\, t)$ (at any instant in time), where s is the parameter which maps out the curve, then the streamlines are the solutions of

$$\frac{d\mathbf{x}}{ds} \propto \mathbf{u} \quad \text{or} \quad \frac{d\mathbf{x}}{ds} = \mathbf{u}(\mathbf{x}, t) \quad \text{(at fixed } t\text{)}. \tag{1.17}$$

In this second representation, the constant of proportionality has been absorbed into the definition of s. Then, for example, in rectangular Cartesian coordinates this vector equation becomes the three scalar equations

$$\frac{dx}{ds} = u, \quad \frac{dy}{ds} = v, \quad \frac{dz}{ds} = w,$$

or equivalently,

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}, \tag{1.18}$$

for the streamlines. (The streamline should not be confused with the *path* of a particle; this is defined (see Q1.4 and also Q1.19) by

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}, t), \tag{1.19}$$

so particle paths and streamlines coincide, in general, only for steady flow; see Q1.19.) The streamlines provide a particularly effective way of describing a flow field: even a simple sketch of the streamlines for a flow often enables important characteristics to be recognised at a glance. (An associated concept, the *stream function*, is described in Q1.20–Q1.23.)

We now turn to a brief consideration of the results that can be obtained when the vorticity, $\boldsymbol{\omega}$, is introduced into Euler's equation, (1.12),

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla P + \mathbf{F}. \tag{1.20}$$

For our purposes we shall assume that $\rho =$ constant (but see Q1.18), and that the body force is represented by a *conservative* force field: $\mathbf{F} = -\nabla\Omega$ for some *potential function* $\Omega(\mathbf{x}, t)$, where the negative sign is a convenience. (This choice for $\mathbf{F}$ applies to most examples of interest; for our studies we shall use $\Omega = gz$ where g is the (constant) acceleration of gravity and z is measured positive upwards.) Equation (1.20) therefore becomes

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla\left(\frac{P}{\rho} + \Omega\right),$$

which is rewritten by introducing the identity (see Q1.1)