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978-0-521-59162-1 - Dynamical Systems and Semisimple Groups: An Introduction
Renato Feres
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126 Dynamical Systems and Semisimple Groups

The theory of dynamical systems can be described as the study of the global properties of groups of transformations. The historical roots of the subject lie in celestial and statistical mechanics, for which the group is the time parameter. In some of its recent developments, the theory is concerned with the dynamics of more general, bigger groups than the additive group of real numbers, particularly semisimple Lie groups and their discrete subgroups. Some of the most fundamental discoveries in this area are due to the work of G. A. Margulis and R. Zimmer. This book comprises a systematic, self-contained introduction to the Margulis–Zimmer theory and provides an entry into current research.

Taking as prerequisites only the standard first-year graduate courses in mathematics, the author develops in a detailed and self-contained way the main results on Lie groups, Lie algebras, and semisimple groups, including basic facts normally covered in first courses on manifolds and Lie groups plus topics such as integration of infinitesimal actions of Lie groups. He then derives the basic structure theorems for the real semisimple Lie groups, such as the Cartan and Iwasawa decompositions, and gives an extensive exposition of the general facts and concepts from topological dynamics and ergodic theory, including detailed proofs of the multiplicative ergodic theorem and Moore's ergodicity theorem.

This book should appeal to anyone interested in Lie theory, differential geometry, and dynamical systems.

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To my parents

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Preface

An action of a group G on a set M is a homomorphism, $g \to \Phi_g$, from G into the group of invertible transformations of M. Traditionally, actions of $\mathbb{R}$ and $\mathbb{Z}$ have been the main objects of concern of the theory of dynamical systems. For example, if X is a smooth vector field on a compact manifold M, its flow defines an action of $\mathbb{R}$ on M. One thinks of the group as parametrizing "time," so that the orbit

$$G \cdot x := \{\Phi_g(x) \mid g \in G\}$$

describes the evolution of the system, starting from an initial state represented by $x \in M$.

For actions of a more general group G, the "time evolutions" associated to its various one-parameter subgroups are "interlocked" according to the algebraic structure of G. Thus, suppose that instead of a single smooth vector field on M we have a family of them, $X_1, \ldots, X_m$, whose Lie brackets satisfy

$$[X_i, X_j] = \sum_{k=1}^{m} a_{ij}^k X_k,$$

where the a_{ij}^k are constants. This means that these fields span a finite-dimensional Lie algebra, associated to a Lie group G. By Lie's second fundamental theorem (essentially theorem 3.9.8), these fields integrate to an action of (the universal covering group of) G. If, for example, the constants a_{ij}^k vanish, G is an abelian group, and the flows of $X_1, \ldots, X_m$ may be thought to define "noninteracting" evolutions.

On the opposite extreme to abelian groups are the semisimple groups. A – not very revealing – definition of semisimplicity is that the matrix (c_{ij}), with entries given by

$$c_{ij} = \sum_{k=1}^{m} \sum_{l=1}^{m} a_{il}^k a_{kj}^l,$$

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is nonsingular. The various subgroups of a semisimple group are tightly interwoven, and one might expect the dynamical properties of actions of G to be accordingly constrained.

The actions considered in this book will, for the most part, be assumed to possess a finite invariant measure. Invariance of a measure μ means that for each measurable subset $A \subset M$ we have $\mu(\Phi_g(A)) = \mu(A)$ for each $g \in G$. The existence of a finite invariant measure forces upon the long-term evolution of the system a kind of statistical regularity, given by the ergodic theorems of chapters 8 and 9, most of which are for actions of $\mathbb{R}$ or $\mathbb{Z}$. This is the basic setup of ergodic theory. Actions of $\mathbb{R}$ and $\mathbb{Z}$ very commonly admit (possibly singular) invariant measures – this is always the case if the action is continuous and M is compact. For nonabelian groups, however, this is a strong requirement.

One can obtain much useful information about a group action on M (assuming that the action is differentiable) by studying its linearization along the orbits. A good understanding of this linearization can yield information about the global properties of the system. (For $\mathbb{Z}$-actions, a modest attempt to justify this claim will be made at the end of chapter 9, in a brief discussion on Pesin theory.) One of the main results of the book can be formulated as follows: If the $\mathbb{Z}$-action is part of a differentiable action by a noncompact semisimple Lie group G on an n-dimensional manifold M, preserving a finite measure μ, then it is possible to give a very precise description of the linearization, along almost every orbit (relative to μ), in terms of the representations of G in dimension n. The key result behind this vague assertion is Zimmer's cocycle superrigidity theorem, which will be studied in chapter 10.

The ergodic theory of actions of semisimple Lie groups and their discrete subgroups has grown in the past decade into a large and very active chapter of the general theory of dynamical systems. It is also a subject with deep foundations; most notably, the work by G. A. Margulis concerning rigidity and arithmeticity of lattices in semisimple groups and the work of R. Zimmer, in particular his cocycle superrigidity theorem. The main purpose of this book is to serve as a relatively gentle introduction to the rigidity theorems of Margulis and Zimmer.

Passing from some knowledge of the linearization of the system along orbits to an understanding of its global topological structure is a hard and wide-open problem. R. Zimmer conjectured in [37] that the actions of Lie groups and lattices considered in this book should be, in some sense, "classifiable" on the basis of a few classes of well-understood examples. This classification program gained momentum with the introduction of ideas from hyperbolic dynamics and the theory of Anosov diffeomorphisms, by S. Hurder, A. Katok, and J. Lewis, about six or seven years ago, and continues today with great vitality. In spite of

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all the recent progress, it is clear that this is not an area of research approaching exhaustion any time soon.

Although I make no attempt to survey the current research activity, the book does contain a few new ideas. The proof of the cocycle superrigidity theorem given in chapter 10 is new, and is due to F. Labourie and myself [11]. The presentation is also different from that in [36] in that it uses throughout a differential-geometric language that some readers may find more natural, or at least more congenial, than the language of cocycles over a group action. I hope that the experts will see some novelty here. In any event, the book was written having in mind primarily the nonexpert, especially the graduate student interested in Lie theory and dynamical systems.

The reader is assumed to have a good working knowledge of measure and integration and the basic theory of differentiable manifolds. Even though first courses on manifolds usually provide some acquaintance with Lie groups and Lie algebras, I have chosen to develop this subject in detail and from the beginning, up to those results that are needed for chapters 8, 9, and 10.

The text, in effect, integrates two courses in one. First, it contains an introduction to part of the modern theory of dynamical systems. This comprises chapters 1, 2, 8, and 9. The dynamics "subcourse" culminates with a detailed proof of the multiplicative ergodic theorem of Oseledec and a hurried discussion of Pesin theory. Taken in isolation, this is necessarily a lopsided account of dynamics. For an even-handed and thorough introduction to dynamical systems, the reader cannot presently do better than to go to [16]. The second "subcourse" is on Lie groups, Lie algebras, and semisimple groups, and comprises chapters 3, 5, and 7. The reader who wishes to pursue this topic further will have no difficulty in continuing on with, say [17], from the point where chapter 7 ends.

There is a large degree of independence among chapters 1 through 9, although almost everything that is developed in them is put to use in chapter 10. The main exceptions are chapter 6, which relies on facts about algebraic actions discussed in chapter 4; section 9.3, which uses some of the language introduced in section 6.1; and section 8.3 on Moore's ergodicity theorem, which uses some of the structure theorems for semisimple Lie groups. Chapter 4 is independent of the first three, although it is also the least self-contained in the book. It consists of a very pedestrian introduction to algebraic geometry and algebraic actions.

The shortest path to arrive at the main results of the book, namely, theorems 10.4.1, 10.6.1, and 10.6.2, is to read chapters 4, 6, and 7, sections 8.2 and 8.3, and the first sections of 10, referring to the earlier chapters for the definition of some occasional unfamiliar term. All the other chapters are there to provide a dynamical systems "context" for the results of chapter 10.

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It seemed to me appropriate from the point of view of the "narrative" to place chapter 6, on geometric structures, immediately after a discussion on the classical groups, even though the results from that chapter are not needed until chapter 10. The reader should keep this in mind in case the discussion in chapter 6 seems at first too formal and unmotivated.

The exercises are an integral part of the text. Their main purpose is to expand or illustrate an idea under discussion. Occasionally, an exercise may also be referred to in proofs. They should always be read, even if not always worked on.

The book began as a short series of lectures given at Penn State University in 1996, and was later expanded after a one-semester course I taught at Washington University, in St. Louis, in 1997. I am grateful to Anatoly Katok for inviting me to give the lectures at Penn State and for bringing my notes to the attention of Cambridge University Press. Had I pursued the subject just a little further the impact of his own work would also become apparent in the text. I am also especially indebted to Robert Zimmer, from whom I learned much of this subject firsthand; to François Labourie, my collaborator in [11], the reference on which much of chapter 10 is based; to Scot Adams, for many enlightening conversations and, in particular, for explaining to me the proof of Moore's ergodicity theorem given in chapter 8; and to Mohan Kumar, Mark de Cataldo, and Vladimir Masek, for lending their expertise on algebraic geometry. Finally, I would like to thank Peter Lampe, Michelle Penner, Holly Lowy, Lawrence Roberts, Mark de Cataldo, and Meeyoung Kim for suggesting many improvements and corrections to the original manuscript. Thanks to them, many – but certainly not all – embarrassing mistakes, obscure phrases, wrong signs, and barbarisms of the first draft will not appear in print.

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