1

Topological Dynamics

The theory of dynamical systems, loosely speaking, studies those properties of group actions that are asymptotic in nature, that is, that become apparent as we "go to infinity" in the group. We call a set X equipped with an action of a group G a *dynamical system* with group G or, alternatively, a G*-space*.

After introducing a few notions that apply to general group actions, we focus our attention on some of the basic properties of topological G-spaces. Other aspects of dynamical systems, relating, for example, to their measurable or smooth properties, will be discussed in later chapters.

1.1 *G*-Spaces

Let G be a group and X a set. A G*-action* on X is a map $\Phi : G \times X \to X$ that satisfies the following two properties:

1. $\Phi(e, x) = x$ for all $x \in X$, where e is the identity of G.
2. $\Phi(g_2, \Phi(g_1, x)) = \Phi(g_2 g_1, x)$ for all $g_1, g_2 \in G$ and $x \in X$.

For each $g \in G$, let $\Phi_g : X \to X$ be defined by $\Phi_g(x) := \Phi(g, x)$. Then Φ_g is a bijection from X onto itself, with inverse $\Phi_{g^{-1}}$, and the map $g \mapsto \Phi_g$ from G into the group of bijective self-maps of X is a group homomorphism. We often write $g \cdot x$ or $g(x)$, or simply gx, instead of $\Phi(g, x)$. The definition of G-action just given is usually called a *left action* of G. By a *right action* of G on M we mean a map $\Phi : M \times G \to M$ such that property 2 is replaced with

$$\Phi(\Phi(x, g_1), g_2) = \Phi(x, g_1 g_2).$$

For each $x \in X$, we define the *orbit* of x by

$$Gx := \{\Phi_g(x) \mid g \in G\}.$$

The orbits of a G-action partition X into disjoint sets; namely, the Gx are the equivalence classes of the relation

$$x \sim y \text{ if and only if there exists } g \in G \text{ such that } x = gy.$$

The *orbit space* is the set of equivalence classes, denoted $G\backslash X$. The action Φ is called *transitive* if the G-space has only one orbit, that is, $X = Gx$ for some x.

Typically, the G-action will leave invariant, or preserve, some structure on X such as a topology, a measurable structure, a smooth manifold structure, or an algebraic variety structure. Of course, these structures are not independent. For example, when studying a smooth group action on a compact Riemannian manifold whose volume form is invariant under the action, one could find it useful at times to focus, say, on the underlying measure-space structure determined by the Borel-measurable sets and the measure obtained by integrating the volume form. The group G, however, will always be regarded here as being, at least, a *topological group*, that is, a Hausdorff space that is also an abstract group for which multiplication and inversion are defined by continuous maps.

A *topological G-space* consists of a topological space X and a continuous action Φ of G on X. In this case each Φ_g, $g \in G$, is a homeomorphism of X. Some of the basic properties of topological G-spaces are discussed in this chapter.

A *smooth G-space* consists of a smooth (C^∞) manifold X and a smooth action Φ of a Lie group G. In this case, each Φ_g is a C^∞ diffeomorphism of X. We discuss smooth actions in chapter 3.

A *measurable G-space* consists of a measurable space $(X, \mathcal{B})$, where $\mathcal{B}$ is a σ-algebra of subsets of X, and a measurable action $\Phi : G \times X \to X$. We will often be interested in actions that preserve a finite measure μ on $(X, \mathcal{B})$. In that case, μ can be normalized so that $\mu(X) = 1$, and μ is then called a *probability measure*. The study of group actions on measurable spaces is the subject of *ergodic theory*, to which we return later, beginning in the next chapter.

An *algebraic G-space* consists of an algebraic variety X defined over some field k (which, in this book, will almost always be $\mathbb{R}$ or $\mathbb{C}$) and an algebraic group G and an algebraic action Φ, both defined over k. Algebraic actions will play an important role in some of the results described in this book. Definitions and general properties concerning them are discussed in chapter 4.

The remainder of the chapter concentrates on the elementary properties of *topological dynamical systems*, that is, topological G-spaces.

Let H be a closed subgroup of G. Then the coset space

$$G/H = \{gH \mid g \in G\}$$

has the quotient topology induced by the natural projection $\pi : G \to G/H$, $\pi(g) = gH$; namely, the open subsets of G/H are $\pi(U) = \{gH \mid g \in U\}$ for

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all open sets $U \subset G$. With respect to the quotient topology, π is continuous and open and G/H is a Hausdorff space.

A *discrete subgroup* of G is a subgroup that is a discrete subset in the topology of G. If G is connected and locally arcwise connected and H is a closed subgroup, it can be shown that H is a discrete subgroup if and only if π is a covering map. If H is discrete and G/H is compact, we say that H is a *uniform lattice* of G.

The *kernel* of an action Φ, denoted $\mathrm{Ker}(\Phi)$, is the kernel of the homomorphism $g \mapsto \Phi_g$, which is a normal subgroup of G. When $\mathrm{Ker}(\Phi)$ is trivial, the action is said to be *effective*. If the action is not effective, Φ induces an effective action of $G/\mathrm{Ker}(\Phi)$ on X. The action is called *locally effective* if $\mathrm{Ker}(\Phi)$ is a discrete subgroup of G.

For each $x \in X$, the *isotropy group of* x is defined by

$$G_x := \{g \in G \mid gx = x\}.$$

G_x is a subgroup of G and it is immediate that $G_{gx} = gG_xg^{-1}$ for each $g \in G$ and $x \in X$. Moreover, $\mathrm{Ker}(\Phi) = \bigcap_{x \in X} G_x$. If $G_x = \{e\}$ for all $x \in G$, we say that the G-action is *free*. The action is called *locally free* if G_x is a discrete subgroup of G for all x in X.

Recall that a topological space X is said to be T_1 if each point $x \in X$ is closed. It is an easy consequence of the definitions that, whenever X is a T_1 G-space, each isotropy group G_x as well as the kernel of Φ are closed subgroups of G and that $G/\mathrm{Ker}(\Phi)$ is a topological group in a natural way. Moreover, the induced (effective) action of $G/\mathrm{Ker}(\Phi)$ makes X a topological $G/\mathrm{Ker}(\Phi)$-space.

1.2 The Orbit Space

Until we impose any further requirements G will be a locally compact second countable topological group and X a complete second countable metrizable G-space. We give $G\backslash X$ the quotient topology induced by the natural projection that to each $x \in X$ associates its orbit.

It will be apparent from some of the examples described later that the orbit space $G\backslash X$ can easily fail to have good separation properties, due to the existence of orbits that wander about in X in a complicated way. This is not the case, however, when Φ is a *proper action*. By definition, Φ is a proper action if for each $x, y \in X$ there exist neighborhoods U of x and V of y such that

$$\{g \in G \mid V \cap gU \neq \emptyset\}$$

is relatively compact. Clearly, the action is proper whenever G is compact.

Exercise 1.2.1 Show that the orbit space $G\backslash X$ of a proper action is Hausdorff. In particular, each orbit Gx is closed in X. Also show that, for each $x \in X$, the map $\phi_x : G/G_x \to Gx$, $gG_x \mapsto gx$, is a homeomorphism.

A somewhat more complicated situation, but still rather simple from the viewpoint of the general theory of dynamical systems, corresponds to the case in which the σ-algebra $\mathcal{B}$ of Borel sets, that is, the σ-algebra generated by the open sets in $G\backslash X$, is *countably separating*. This means that there is a sequence $B_i \in \mathcal{B}$ such that for each pair of points in X one can find a B_i that contains exactly one of the two points. In this case, the G-action will be called *tame*. Notice that a proper action is tame. In fact, the quotient topology of $G\backslash X$ is second countable, since X is second countable, and Hausdorff, so points can already be separated by open sets.

The next theorem gives a useful characterization of tame actions. It is taken from [36], where a tame action is called *smooth*. The result is due to Glimm and Effros. The orbit Gx of a topological G-space X is *locally closed* if it is open in its closure $\overline{Gx} \subset X$.

Theorem 1.2.2 Suppose that Φ is a continuous action of a locally compact second countable group G on a complete second countable metrizable space X. Then the following are equivalent:

1. All orbits are locally closed.
2. The action is tame.
3. For every $x \in X$, the natural map $G/G_x \to Gx$ is a homeomorphism, where Gx has the relative topology as a subset of X.

Proof. The implication $2 \Rightarrow 1$ is the hardest to prove and will not be discussed here. We refer the reader to [36, 2.1.14] for a proof. We begin with the assertion $1 \Rightarrow 2$. Since the topology of X has a countable basis and the projection $\pi : X \to G\backslash X$ is open, the topology of $G\backslash X$ also has a countable basis. To prove that the Borel-measurable structure is countably separating it suffices to show that $G\backslash X$ is a T_0-space, that is, that we can separate any two points by an open set that contains only one of the points. Let $x, y \in X$. If $\pi(x)$ and $\pi(x)$ are not separated by an open set, $Gy \subset \overline{Gx}$ and $Gx \subset \overline{Gy}$. Therefore Gy is dense in $\overline{Gx}$. But by assumption Gx is open in its closure, so $Gy \cap Gx \neq \emptyset$. This implies that $\pi(x) = \pi(y)$.

We now show that 3 and 1 are equivalent. We may assume without loss of generality that Gx is dense in X. If this is not the case, simply let X denote the closure of that orbit. We begin with $3 \Rightarrow 1$ and assume that $G/G_x \to Gx$ is a homeomorphism. Then Gx with the subspace topology satisfies the Baire category theorem, because G/G_x satisfies it. (G is locally compact, hence a Baire space. It follows that the quotient is also a Baire space.) Now, G is σ-compact, being second countable and locally compact. Therefore, by Baire's theorem, some compact set $A \subset Gx$ contains a nonempty open set, that is, for some nonempty open set $U \subset X$, $U \subset U \cap \overline{Gx} \subset A$. Thus $Gx = GU$, which is open.

For the converse, suppose that Gx is open in X. Notice that $G/G_x \to Gx$ is continuous, so it suffices to prove that it is also open. We call $U \subset G$ a *symmetric set* if $g \in U$ implies $g^{-1} \in U$. Any open neighborhood V of e contains a symmetric neighborhood: $V \cap V^{-1}$. We claim that it suffices to show that for any compact symmetric set $U \subset G$ whose interior is an open neighborhood of $e \in G$, Ux contains a nonempty open set. Namely, let N be any neighborhood of e and choose a compact symmetric U with $U^2 \subset N$. If Ux contains a neighborhood of some ux, $u \in U$, then $u^{-1}Ux$ contains a neighborhood of x, and hence so does Nx. Therefore $G/G_x \to Gx$ is an open map. To show that Ux contains an open set, choose a countable dense set $\{g_i\} \subset G$. Then $Gx = \bigcup_i g_i Ux$, a union of compact sets, so by the Baire category theorem, we have that one $g_i Ux$ contains an open set, and hence so does Ux. □

Exercise 1.2.3 Show that the $\mathbb{R}$-action on $\mathbb{R}^2$ defined by $\Phi_t(x, y) := (e^t x, e^{-t} y)$, where $t \in \mathbb{R}$ and $(x, y) \in \mathbb{R}^2$, is tame but not proper. Verify that the orbits are locally closed, but the orbit space is not Hausdorff.

The following definitions are concerned with different orbit types. An element x in a G-space X is said to be a *fixed point* if $G_x = G$. It is a *periodic point* if G/G_x is compact. A (topological) G-space X is said to be *topologically transitive* if some G-orbit is dense in X. If *all* orbits are dense, the action is called *minimal*. A subset $A \subset X$ is called *G-invariant* if for each $x \in A$ and $g \in G$, $gx \in A$. An equivalent definition of minimal action is that X does not have a proper closed G-invariant set, since the closure of a G-invariant set is a G-invariant set. A point x of a topological G-space X will be called *recurrent* if for each neighborhood U of x and each compact $K \subset G$, there is g in the complement of K such that $gx \in U$. It is immediate from the definitions that periodic points are recurrent. Furthermore, if both the orbit of x and its complement are dense in X, then x is a recurrent point. We leave the verification of this last claim as an exercise to the reader. Notice that the action of G on itself by translations is topologically transitive – in fact, transitive – but not recurrent.

The preceding notions can all be illustrated with actions defined on the n-torus. Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ denote the n-dimensional torus, defined as the quotient of the abelian group $\mathbb{R}^n$ by its integer lattice subgroup $\mathbb{Z}^n$. The element $x = v + \mathbb{Z}^n$ will also be denoted $[v]$. $\mathbb{T}^n$ can alternatively be described as the product of n copies of the circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$; namely,

$$(a_1, \ldots, a_n) + \mathbb{Z}^n \mapsto (e^{2\pi i a_1}, \ldots, e^{2\pi i a_n})$$

is a diffeomorphism between $\mathbb{T}^n$ and $S^1 \times \cdots \times S^1$.

$\mathbb{R}^n$ acts transitively on $\mathbb{T}^n$ via the smooth action $\Phi : \mathbb{R}^n \times \mathbb{T}^n \to \mathbb{T}^n$ such that

$$\Phi : (u, [v]) \mapsto [u + v].$$

For each $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$ define the translation $\tau_u = \Phi(u, \cdot)$. Then τ_u generates a $\mathbb{Z}$-action on $\mathbb{T}^n$ by $(m, [v]) \mapsto \tau_u^m([v])$. If the components of u are rational numbers, the orbit of each x is periodic and all orbits are finite with same cardinality, as one can easily check.

Real numbers $x_1, \ldots, x_s$ are called *rationally independent* if given integers k_i such that $\sum_{i=1}^{s} k_i x_i = 0$, then $k_i = 0$ for all i.

Proposition 1.2.4 Fix a vector $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$ and consider the $\mathbb{Z}$-action on $\mathbb{T}^n$ generated by τ_u. Then the following statements are equivalent:

1. The action is topologically transitive.
2. The action is minimal.
3. The numbers $1, u_1, \ldots, u_n$ are rationally independent.

Proof. The proof is taken from [16]. It is clear that $2 \Rightarrow 1$. On the other hand, if some $x \in \mathbb{T}^n$ has a dense $\mathbb{Z}$-orbit, then all points have dense orbits since we can get from x to any other point by a translation and all translations commute with the $\mathbb{Z}$-action. Therefore, 1 and 2 are equivalent.

We now show $1 \Rightarrow 3$. Notice that if $1, u_1, \ldots, u_n$ are not rationally independent, we can find integers k_i, not all 0, such that $\sum_{i=1}^{n} k_i u_i = k_0$. Therefore, the function

$$\varphi(v) := \sin\left(2\pi \sum_{i=1}^{n} k_i v_i\right)$$

is a continuous $\mathbb{Z}$-invariant function on $\mathbb{T}^n$, that is, $\varphi \circ \tau_u^m = \varphi$ for all $m \in \mathbb{Z}$. But φ is not a constant function, so there exists $c \in \mathbb{R}$ such that the sets $U = \varphi^{-1}(\{t \in \mathbb{R} \mid t > c\})$ and $V = \varphi^{-1}(\{t \in \mathbb{R} \mid t < c\})$ are nonempty and disjoint. Furthermore, U and V are invariant sets since φ is $\mathbb{Z}$-invariant. It follows that the action cannot be topologically transitive.

If the action is not topologically transitive, there is a nonempty $\mathbb{Z}$-invariant open set U such that $\bar{U} \neq \mathbb{T}^n$. In fact, if no such U exists, one obtains a dense orbit as follows. Let $U_1, U_2, \ldots$ be a countable base of open sets for the topology of $\mathbb{T}^n$. By assumption, there exists an integer N_1 such that $\tau_u^{N_1}(U_1) \cap U_2 \neq \emptyset$. Let V_1 be a nonempty open set such that $\bar{V}_1 \subset U_1 \cap \tau_u^{-N_1}(U_2)$. There exists an integer N_2 such that $\tau_u^{N_2}(V_1) \cap U_3 \neq \emptyset$. Again, take an open set V_2 such that $\bar{V}_2 \subset V_1 \cap \tau_u^{-N_2}(U_3)$. By induction, we construct a nested sequence of open sets V_n such that $\bar{V}_{n+1} \subset V_n \cap \tau_u^{-N_{n+1}}(U_{n+2})$. The intersection $V = \bigcap_{n=1}^{\infty} \bar{V}_n = \bigcap_{n=1}^{\infty} V_n$ is nonempty since the $\bar{V}_n$ are compact. If $x \in V$, then $\tau_u^{N_{n-1}}(x) \in U_n$ for each $n \in \mathbb{N}$. This shows that the orbit of any $x \in V$ intersects each open set in a basis for the topology of $\mathbb{T}^n$. Therefore, the $\mathbb{Z}$-orbit of x is dense.

The previous claim can now be used to show $3 \Rightarrow 1$. Thus, suppose that the action is not topologically transitive, which implies by the claim that there exists an open nonempty $\mathbb{Z}$-invariant set U whose closure is not $\mathbb{T}^n$. Let χ be the characteristic function of U. In what follows, we think of χ as a function on $\mathbb{R}^n$ that is periodic in each variable. Since U is invariant, we have $\chi \circ \tau_u = \chi$. Take the Fourier expansion

$$\chi(x_1, \ldots, x_n) = \sum_{k \in \mathbb{Z}^n} c_k e^{2\pi i k \cdot x},$$

where $k \cdot x$ denotes the ordinary dot product $k \cdot x = k_1 x_1 + \cdots + k_n x_n$. Then

$$\chi(\tau_u(x)) = \sum_{k \in \mathbb{Z}^n} c_k e^{2\pi i k \cdot u} e^{2\pi i k \cdot x}.$$

Invariance of χ and uniqueness of the Fourier expansion imply $c_k = c_k e^{2\pi i k \cdot u}$ for each $k \in \mathbb{Z}^n$. If $c_k = 0$ for all nonzero $k \in \mathbb{Z}^n$, it would follow that χ is constant almost everywhere with respect to the Lebesgue measure on $\mathbb{R}^n$. Therefore, the measure of either U or its complement would be 0, which is not the case. Therefore, for some nonzero $k \in \mathbb{Z}^n$, $c_k \neq 0$, whence $e^{2\pi i k \cdot u} = 1$. This shows that the numbers $1, u_1, \ldots, u_n$ are rationally dependent. □

Exercise 1.2.5 Show that if X is a locally compact second countable space and $\Phi : G \times X \to X$ is a topological action, then Φ is topologically transitive if and only if any two nonempty open G-invariant sets intersect. (The argument is essentially contained in the preceding proof.) Also show that if X is a Baire space (e.g., locally compact) and the action is topologically transitive, then the set of points with a dense orbit is a dense G_δ-set, that is, it is a countable intersection of open dense sets.

Orbits of different points of a G-space can be very different, as the next example will show. Let $SL(n, \mathbb{Z})$ be the group of n-by-n matrices of determinant

1 with integer entries. Since the linear action of $SL(n,\mathbb{Z})$ on $\mathbb{R}^n$ leaves invariant the integer lattice $\mathbb{Z}^n$, we obtain an action on the torus

$$\Phi : SL(n,\mathbb{Z}) \times \mathbb{T}^n \to \mathbb{T}^n$$

by setting $\Phi(A,[v]) := [Av]$, where Av denotes matrix multiplication of A and $v \in \mathbb{R}^n$, the latter now viewed as a column vector. The next exercise shows that Φ is topologically transitive but not minimal.

Exercise 1.2.6 Show that each point $[u_1,\ldots,u_2] \in \mathbb{T}^n$ with rational components is a periodic point for the above action of $SL(n,\mathbb{Z})$ on $\mathbb{T}^n$. Using the argument employed in the last proposition, show that the action is topologically transitive. The same argument can be used to show that the $\mathbb{Z}$-action on $\mathbb{T}^2$ generated by the single matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ is also topologically transitive.

The example introduced in the previous exercise can be generalized as follows. Let G be a topological group and Γ a discrete subgroup of G such that the quotient $X = G/\Gamma$ is compact or, more generally, such that X admits a G-invariant probability measure. (Invariant measures will be defined and discussed in detail in a later chapter.) Let H be a closed noncompact subgroup of G and define an H-action on X by

$$\Phi(h, g\Gamma) := hg\Gamma.$$

It will follow from results of chapter 8 that if $G = SL(n,\mathbb{R})$ (or any other connected, noncompact, simple Lie group) then for any noncompact closed subgroup H of G, the H-action on X is topologically transitive.

1.3 Suspensions

Starting with an action $\Phi : \Gamma \times X \to X$, where Γ is a discrete subgroup of a connected topological group G, it is possible to define in a canonical way a locally free action $\bar{\Phi}$ of G that "looks transversely like" Φ, called the *suspension* of Φ, or the *induced action* from Φ. It is defined as follows. First notice that Γ acts diagonally on the product $G \times X$ by

$$\gamma \cdot (g, x) = (g\gamma^{-1}, \Phi(\gamma, x)).$$

We denote the orbit space by $S = (G \times X)/\Gamma$ and the element of S represented by (g, x) will be written $[g, x]$.

S is the total space of a fiber bundle $p : S \to G/\Gamma$, $(g, x)\Gamma \mapsto g\Gamma$ (see the beginning of chapter 6 for the definition of a fiber bundle), whose fibers are homeomorphic to X. In fact, let $\pi : G \to G/\Gamma$ be the natural projection (a covering map) and, for each $z \in G/\Gamma$, let U be a sufficiently small connected neighborhood of z such that $\pi^{-1}(U)$ is a disjoint union of open sets in G, each homeomorphic to U via π. Choose one component of the preimage, say U_0, and set $\sigma := (\pi|_{U_0})^{-1} : U \to U_0$. Then

$$(z, x) \mapsto [\sigma(z), x]$$

is a homeomorphism between $U \times X$ and $p^{-1}(U)$, showing local triviality. Notice that by making a different choice of connected component of $\pi^{-1}(U)$, say $U_1 = U_0\gamma$, then the change of trivialization is given by the map from $U \times X$ onto itself that sends (z, x) to $(z, \Phi(\gamma, x))$ for some $\gamma \in \Gamma$ independent of z.

The group G acts on S by

$$\bar{\Phi}(h, [g, x]) := [hg, x].$$

Notice that $[hg, x] = [hg\gamma^{-1}, \Phi(\gamma, x)]$, so the action is indeed well defined and is clearly continuous.

For example, let $G = \mathbb{R}$ and $\Gamma = \mathbb{Z}$, and consider the $\mathbb{Z}$-action on $\mathbb{T}^n$ generated by a diffeomorphism τ of the n-torus. Then $G/\Gamma = \mathbb{R}/\mathbb{Z} = \mathbb{T}^1$, so the suspension of the $\mathbb{Z}$-action is an $\mathbb{R}$-action on $\mathbb{T}^{n+1}$.

Similarly, one obtains an $SL(n, \mathbb{R})$-action on a fiber-bundle with typical fiber $\mathbb{T}^n$ over $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$, by suspending the $SL(n, \mathbb{Z})$-action on $\mathbb{T}^n$ defined earlier.

Exercise 1.3.1 Show that the Γ-action Φ on X is topologically transitive if and only if the suspension $\bar{\Phi}$ is topologically transitive.

1.4 Dynamical Invariants

Two topological G-spaces X and Y are said to be *(topologically) equivalent* if there exists a homeomorphism $F : X \to Y$ that intertwines the respective G-actions. More precisely, $F(gx) = gF(y)$ for each $x \in X$, $y \in Y$, and $g \in G$. Equivalent G-spaces have the same (topological) dynamical properties; for example, the G-action on X is topologically transitive if and only if the action on Y is, and periodic orbits of X correspond under F to periodic orbits of Y.

Any attempt to classify topological G-spaces in terms of their global dynamical properties immediately calls for characteristic "quantities" that have the potential to distinguish inequivalent systems. An analogy can be made with linear maps of vector spaces. Linear maps $T_i : V_i \to V_i$, $i = 1, 2$, are equivalent if there exists a linear isomorphism $F : V_1 \to V_2$ such that $FT_1 = T_2F$.

Equivalent linear maps must have the same spectrum of eigenvalues, so the spectrum is an "invariant" in this case. (There is no need to define the concept of an invariant formally, since we will be concerned only with some specific examples of them.)

An interesting example of an invariant for G-spaces can be defined as follows. Let $H_q(X_i, \mathbb{R})$ denote the real qth singular homology spaces of G-spaces X_i, $i = 1, 2$. (For the basic definitions in algebraic topology see, for example, [14]. This example, however, will not be needed later.) Then G acts in a natural way on $H_q(X_i, \mathbb{R})$ by vector space isomorphisms. If the actions on X_1 and X_2 are equivalent, via a homeomorphism $F : X_1 \to X_2$, then the linear actions of G on $H_q(X_i, \mathbb{R})$ are linearly equivalent via the map induced by F on the homology spaces, which we denote $H_q(F)$. Therefore, the linear invariants of $H_q(f_i)$ are also invariants of the G-spaces.

We now restrict our attention to $\mathbb{Z}$-actions. Thus, let $f : X \to X$ be a homeomorphism of a topological space X, generating a $\mathbb{Z}$-action on X. Denote by $P_n(f)$ the number of periodic points of f of period n, that is, the cardinality of the set of fixed points of f^n. Define the *exponential growth rate of periodic points* by

$$p(f) := \limsup_{n\to\infty} \frac{1}{n} \log(\max\{P_n(f), 1\}).$$

It is immediate that $p(f)$ is also an example of a topological invariant for $\mathbb{Z}$-actions.

Exercise 1.4.1 Let $f : \mathbb{T}^n \to \mathbb{T}^n$ be a homeomorphism of the n-torus obtained from an integer matrix A of determinant 1. Suppose that none of the eigenvalues of A lie on the unit circle $\{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$. (For example, $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.) Show that

$$p(f) = \sum_{|\lambda|>1} \log|\lambda|,$$

where the sum ranges over the eigenvalues of A of modulus greater than 1. In particular, the number of periodic points of period m grows exponentially with m. (For the details, see [16]. The key point here is to use the Lefschetz fixed point formula, from which one derives

$$P_m(f) = |\det(I - H_1(f)^m)|,$$

where $H_1(f) = A$ is the linear map induced by f on $\mathbb{R}^n = H_1(\mathbb{T}^n)$.)

An important topological invariant of $\mathbb{Z}$-actions is the *topological entropy*. Roughly speaking, it captures the exponential growth rate, as $m \to \infty$, of the number of orbit segments of length m that can be distinguished with a