1

Some useful electrical circuits

Electrical circuits are important in our lives, certainly in electric guitars, television sets, and computers. However, they are also important in understanding how our bodies work. In fact, most nerve activity, including that in the brain, is electrical: Ionic currents passing through membranes and across synaptic gaps are the dominant physical properties of neurons. This first chapter introduces some basic elements of electrical circuits, starting with simple resistors and ending with a description of some modern integrated circuit chips that behave like neurons.

Electrical circuits are described in terms of the physical quantities of voltage (V) and current (I). Voltages and currents are not intuitive; they cannot be directly observed by us without special instruments such as voltmeters and ammeters. However, we can think of voltage as being a pressure that pushes electrons in a conductor and of current as measuring the electron flow. We gain intuition about voltage and current by thinking of them as being solutions of the appropriate mathematical models, either ones derived here from Kirchhoff's Laws for the elementary circuits or, more generally, ones derived from Maxwell's equations in more advanced work.

The circuits studied here involve several components or *circuit elements*. These are listed next along with their notations and *IV* (current-voltage)-characteristics. Circuit elements can be combined to form circuits, and these can be modeled using Kirchhoff's laws. *RLC* circuits are important examples of simple circuits. Other important circuits presented in this chapter include filters and oscillator feedback circuits.

1.1 Circuit elements

Circuits are combinations of physical devices such as resistors and capacitors, and they are described using mathematical terms.

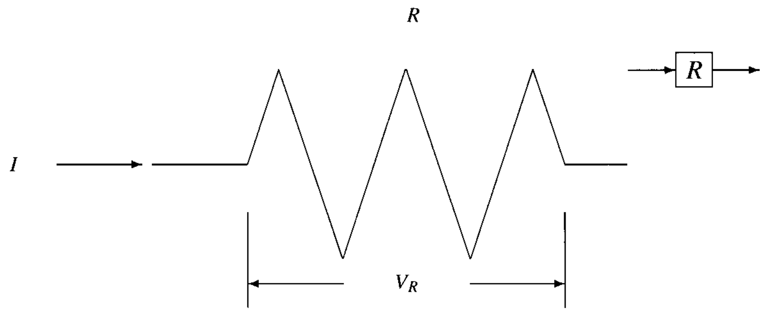

Figure 1.1. A resistor. A current I passing through the resistor with resistance R creates a voltage change across the device of size $V_R = RI$.

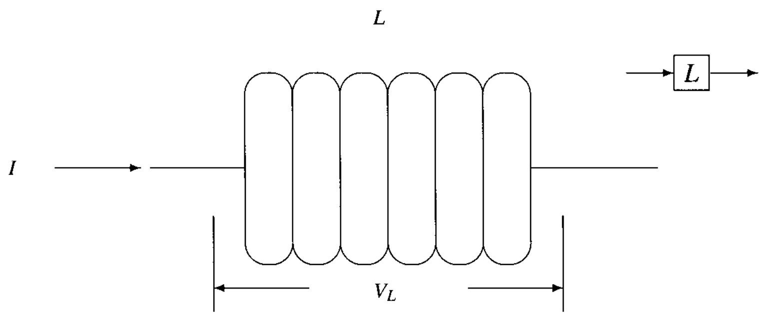

Figure 1.2. An inductor. A current I passing through an inductor creates a voltage change of size $V_L = L\dot{I}$.

Resistors are devices that impede the flow of current. Let I be the current into the resistor and V_R the voltage across it as shown in Figure 1.1. Observations of how V_R and I are related led to *Ohm's law*

$$V_R = RI.$$

That is, the voltage across a resistor is proportional to the current through it. The constant of proportionality, the *resistance* R, is measured in units of ohms. A high (low) resistance with a fixed voltage results in a low (high) current.

Inductors are coils of wire wrapped around a metal core (see Figure 1.2). Current through the coil induces a magnetic field in the core that creates a voltage. In an inductor I and V_L are related by the formula

$$V_L = L\dot{I},$$

where $\dot{I} = dI/dt$. The constant L is called the *inductance*, and it is measured in units of henrys. Inductors are important in circuits described in later chapters,

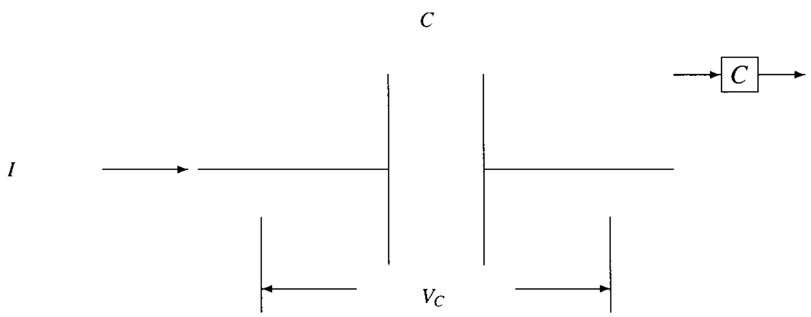

Figure 1.3. A capacitor. Charge accumulates on the plates of the capacitor at a rate proportional to I, and there results a voltage change $\dot{V}_C = I/C$.

but there they are replaced by more convenient integrated circuits. Still, the result has the same *IV*-characteristic and so inductors are written into circuits even though they are replaced by other devices.

A *capacitor* is a device that accumulates charge on plates separated by a nonconductor (Figure 1.3). The charge coming in (I) accumulates and the *IV*-relation is

$$V_C = \frac{1}{C} \int_0^t I \, dt$$

or, equivalently,

$$I = C\dot{V}_C.$$

The constant C is called the *capacitance*, and it is measured in units of farads. In most of the circuits used here the appropriate units are microfarads (10^{-6} farads).

1.1.1 Electromotive force

A power supply, such as a battery or an alternating voltage, applies an electromotive force (voltage) to a circuit. We denote an electromotive force by E and depict it as shown in Figure 1.4. When arrows are present, they indicate the direction from positive to negative for the variables they describe.

1.1.2 Voltage adders and multipliers

Voltages can be added and multiplied by using a combination of operational amplifiers [68]. Figure 1.5 indicates the notation used for a voltage adder and

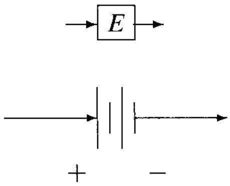

Figure 1.4. A battery. Although it is indicated here, we ignore the polarity of batteries in our subsequent diagrams, but this is corrected for in the signs of currents and voltages in the circuits.

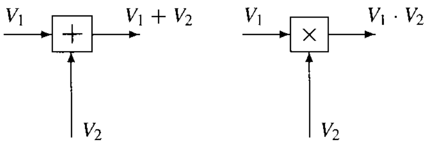

Figure 1.5. A voltage adder (left) and a voltage multiplier (right).

for a voltage multiplier. The result of input voltages u and v is their sum $u + v$ or their product $u \cdot v$, respectively.

1.2 Filters

An important class of circuits involves a resistor, an inductor, and a capacitor in series with a battery. These are called *RLC* circuits. A mathematical description of an *RLC* circuit can be derived using Kirchhoff's laws.

Filters are important since they sort out and allow to pass only certain frequencies. The purpose is often to eliminate noise from a signal or to restrict a signal to a size that meets tolerances of circuit elements farther downstream. The filters described here are used in various applications later.

1.2.1 Kirchhoff's laws

Most circuit models are derived using Kirchhoff's laws. These state that:

- The total voltage measured around any closed loop that can be drawn in the circuit is zero.
- The total current into any circuit node sums to zero.

A circuit node is any point at which two or more wires come together, and a closed loop in a circuit is any closed loop that can be made on a circuit diagram. These definitions are illustrated in the next section.

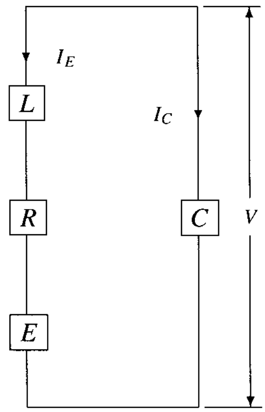

Figure 1.6. An *RLC* circuit.

1.2.2 *RLC circuits*

An *RLC* circuit is shown in Figure 1.6.

The first of Kirchhoff's laws implies that $I \equiv I_E$ satisfies

$$-E + RI + L\dot{I} + V = 0,$$

where the four terms on the left are the applied voltage (measured in the clockwise direction) and the voltages across the resistor, inductor, and capacitor, respectively. The current and the capacitor voltage are related by

$$C\dot{V} = I.$$

Thus, we get a system of two differential equations:

$$C\dot{V} = I,$$
$$L\dot{I} = E - V - RI.$$

These equations can be solved in closed form for V and I once C, L, R, and E are known. This is performed a little later, but first we will consider the geometry of solutions.

1.2.2.1 Geometry of solution of an RLC circuit

Geometric methods play a big role in the study of *RLC* circuits and more complicated circuits. For this, a plot of I versus V is made, and graphs of the solutions to this system of equations are constructed.

The first step is to find *isoclines*. These are the graphs on which $\dot{V} = 0$ and on which $\dot{I} = 0$. These are depicted in Figure 1.7.

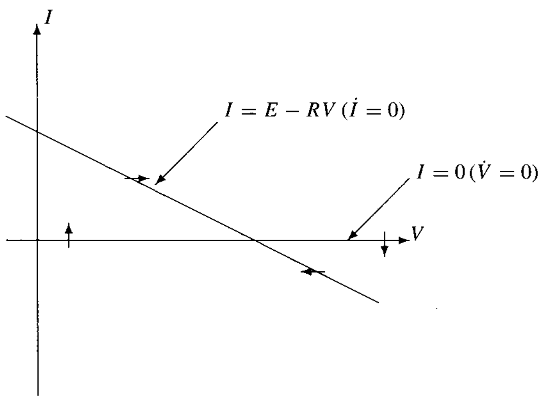

Figure 1.7. Isoclines of the *RLC* circuit.

Obviously, the static states, or equilibria, of the system occur where the isoclines cross: In this case, there is one equilibrium ($I = 0$ and $V = E$). Since $(\dot{V}, \dot{I})$ is a vector tangent to the solutions, we can roughly draw the solutions by observing how they cross the isoclines and axes. Typical crossings are shown in Figure 1.7. It follows that solutions oscillate around the equilibrium.

Next, we determine the solutions in closed form. This is possible because the *RLC* circuit is a linear circuit (see Appendix A).

1.2.2.2 Analytic solution of an RLC circuit

Differentiating the first equation leads to a single second-order differential equation

$$LC\ddot{V} + RC\dot{V} + V = E.$$

This equation can be solved using the Laplace transform method, as shown in Appendix A. Note that when resistance in the circuit is negligible, the model reduces to a harmonic oscillator

$$LC\ddot{V} + V = E,$$

where the natural frequency is shown in Appendix A to be $\sqrt{1/LC}$.

The ratio R/L represents the damping in the full circuit. From the Appendix, we see that if $R > 0$, then solutions spin into the equilibrium. If $R = 0$, then the solutions are ellipses about the equilibrium.

1.2.2.3 LC circuits

RLC circuits with no resistance behave like timers. The circuit shown in Figure 1.8 is referred to as an *LC* circuit.

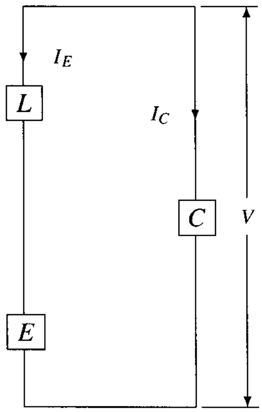

Figure 1.8. A harmonic oscillator.

As shown next, the solutions of this circuit for V and I describe ellipses in the V-I plane centered at $V = E, I = 0$. A radius drawn from this center to the point $(V(t), I(t))$ moves like one of the hands on a clock. In fact, the LC circuit can easily be converted to the frequency domain: Let

$$V = E + r\sin\theta$$

and

$$\sqrt{LC}\dot{V} = r\cos\theta.$$

After some calculation, we obtain

$$\tau\dot{r} = 0$$

$$\tau\dot{\theta}r = r,$$

where the time constant $\tau = \sqrt{LC}$. Therefore, $r =$ constant. If $r \neq 0$, then $\dot{\theta} = 1/\tau$, and equation in the frequency domain completely describes the voltage and current dynamics in the circuit: $V = E + r\sin t/\tau$ and $I = C\dot{V} = r\sqrt{C/L}\cos t/\tau$.

1.2.3 RC circuits; low-pass filters

The circuit in Figure 1.9 describes a low-pass filter where $E = V_{\text{in}}$ is the input voltage and $V = V_{\text{out}}$ is the output voltage.

The mathematical model of a low-pass filter is

$$-V_{\text{in}} + RI + V_{\text{out}} = 0,$$

$$C\dot{V}_{\text{out}} = I.$$

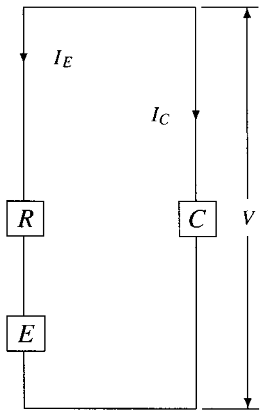

Figure 1.9. A low-pass filter.

This leads to the single equation

$$RC\dot{V}_{\text{out}} + V_{\text{out}} = V_{\text{in}}.$$

Given V_{in}, we can solve this equation to get

$$V_{\text{out}}(t) = V_{\text{out}}(0)e^{-t/RC} + \frac{1}{RC}\int_0^t e^{-(t-s)/RC}V_{\text{in}}(s)\,ds.$$

The last term in this formula is called a convolution integral; in it, past values of V_{in} are weighted and added up.

1.2.4 Transfer functions

An input–output device where the input voltage W is related to the output voltage V by the equation

$$a_n V^{[n]} + a_{n-1}V^{[n-1]} + \cdots + a_0 V = b_m W^{[m]} + \cdots + b_0 W,$$

where $V^{[j]} = d^j V/dt^j$, etc., is referred to as a *general filter*. This equation can be solved using Laplace transforms, as shown in Appendix A. Each derivative is replaced by the transform variable, say s, to the appropriate power: The result is that if $\widetilde{V}(s)$ and $\widetilde{W}(s)$ denote the Laplace transforms of V and W, respectively, then

$$(a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0)\widetilde{V} = (b_m s^m + \cdots + b_0)\widetilde{W} + \text{a polynomial in } s.$$

The last polynomial in s depends on initial conditions, and it is taken to be known from initial conditions. Therefore, the transform of the output voltage is found to be a rational function of s multiplied by $\widetilde{W}$:

$$\widetilde{V}(s) = \widetilde{W}(s)[b_m s^m + \cdots + b_0]/[a_n s^n + \cdots + a_0],$$

where terms involving the initial conditions are ignored. The inverse transform formula gives a convolution integral for V:

$$V(t) = \int_0^t h(t - t')W(t')\,dt' + \text{ a known function,}$$

where h is a function whose Laplace transform is

$$\tilde{h}(s) = [b_m s^m + \cdots + b_0]/[a_n s^n + \cdots + a_0].$$

The function $\tilde{h}$ is called the *transfer function* of the filter, and it is convenient to follow the engineering literature and write

$$V(t) = \tilde{h}(s)W(t)$$

for the convolution integral formula.

Using this notation, we can write the input–output relation for a low-pass filter as

$$V_{\text{out}}(t) = \frac{1}{RCs + 1} V_{\text{in}}(t),$$

which carries all the meaning derived earlier in this section. In particular, this is a convenient shorthand notation for the integral applied to the input voltage.

1.3 Voltage-controlled oscillators (VCOs)

Voltage-controlled oscillators are the basic elements used in some neuron models studied in later chapters. They are electrical oscillators whose frequency is modulated or controlled by an input voltage. There are many kinds of voltage-controlled oscillators available on the electronics market, but we denote a generic one by VCO and depict it by the graph

$$V_{\text{in}} \to \text{VCO} \to V(x(t)),$$

where the input voltage V_{in} and the output voltage V are related in a somewhat complicated way. V is a fixed function, called the wave form (e.g., a square wave, a saw-tooth wave, or a sinusoidal wave). That is, it is a function that is periodic in its argument x and has a fixed shape. All three of these possibilities are available directly from outputs on commercially available VCOs. The function $x(t)$ is the phase of the output voltage. For example, if we select the sinusoidal wave output (a certain pin coming out of the chip) and the device has settled down under steady conditions, then $x(t) = \omega t + \phi$, where ω is the output frequency, ϕ is the phase lead (if positive) or phase lag (if negative), and the output voltage would be $\sin(\omega t + \phi)$.

There exists some confusion in the terminology since sometimes ϕ is referred to as the phase of the output; we will try to be consistent here and describe x as being the phase of $V(x)$ and when appropriate ϕ will be the *phase deviation*.

For example, the potential supplied to households in the United States is 117 volts (root mean square) with a 60-cycle alternating current, and the voltage observed across the terminals in a wall socket is (approximately) $165\cos(2\pi\frac{t}{60} + \phi)$, where time is measured in seconds. The amplitude of the voltage is 165 and its phase is $2\pi\frac{t}{60} + \phi$ (with $\phi = 0, 2\pi/3$, or $4\pi/3$). Here $V(x) = \cos x$ and $x(t) = 2\pi\frac{t}{60} + \phi$. The phase difference, say $\phi_1 - \phi_2$ between two wires in a household, determines the size of the potential drop between the wires.

Current is ignored in VCOs, and the model is given in terms of the input and output voltages alone. So rather than using an *IV*-relation as we did for filters, we describe the device in terms of an input–output relation. The output of the VCO is an oscillatory function V of the phase $x(t)$. When V_{in} is in the operating range of the VCO, the output phase is related to this controlling voltage by the simple differential equation

$$\dot{x} = \omega + \sigma V_{\text{in}},$$

where ω is called the VCO *center frequency* and σ is the VCO sensitivity, both of which are known. Here and below, we take $\sigma = 1$ by suitably scaling input voltages. Keep in mind that the units in this equation are correct although σ does not appear explicitly in the following models.

This equation can be solved by integrating it:

$$x(t) = x(0) + \omega t + \int_0^t V_{\text{in}}(s)\,ds,$$

where $x(0)$ is the initial phase. The output voltage is

$$V\left(\omega t + \int_0^t V_{\text{in}}(s)\,ds + x(0)\right).$$

Thus, the larger is ω or V_{in}, the faster V will oscillate.

We suppose here that all voltages are within the operating range of the VCO device.