1. The Exceptional Set for Goldbach's Problem in Short Intervals

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1. Introduction

Define a *Goldbach number* to be an even number which can be written as the sum of two primes. Ramachandra [26] proved that almost all even numbers in an interval of the form $[x, x+x^\theta]$ are Goldbach numbers, provided that

$$\theta > \tfrac{3}{5}. \tag{1.1}$$

Here and below 'almost all' means 'with less than $x^\theta/\log^A x$ exceptions, provided that $A>0$ and $N > C_1(A,\theta)$.'

Recently there has been a flurry of papers on this problem, by Perelli and Pintz [23], Mikawa [20], Jia [16], [17] and Li [18]. In successive steps the condition (1.1) has been weakened to $\theta > \frac{7}{81}$ (Li, [18]). In the present paper we shall push the method close to what is possible with our existing knowledge of mean and large values of Dirichlet polynomials.

Theorem 1. *For $\theta \ge \frac{11}{160}$, almost all even numbers in $[x, x+x^\theta]$ are Goldbach numbers.*

Remark. We note that $\frac{7}{81} = 0.08641\ldots$ whereas $\frac{11}{160} = 0.06875$.

The idea in all these papers is to show that, for almost all even integers $2n$ in $K = [x, x+x^{\theta_1\theta_2}]$,

$$S(n) := \sum_{\substack{k+m=2n \\ k\in I,\ m\in J}} \rho(k)\rho(m) > 0 \tag{1.2}$$

where ρ is the indicator function of the prime numbers,

$$I = (x-2Y, x] \quad \text{with} \quad Y = x^{\theta_1}, \qquad J = (Y, 2Y].$$

Thus, for example, Perelli and Pintz [23] get an asymptotic formula for $S(n)$ with $\theta_1 \ge \frac{7}{12}+\epsilon$, $\theta_2 \ge \frac{1}{3}+\epsilon$. We shall take

$$\theta_1 = \tfrac{11}{20}+\epsilon, \quad \theta_2 = 2\left(\tfrac{1}{16} - 10^{-5}\right).$$

R.C. Baker is partly supported by a National Science Foundation grant.

Here and below, ϵ is a sufficiently small positive absolute constant. Obviously Theorem 1 follows from (1.2) with this θ_1 and θ_2. In our proof of (1.2) we assume (as we may) that $x - \frac{1}{2}$ is an integer.

To prove (1.2) we borrow a simple but effective inequality from Brüdern and Fouvry [4] (a similar inequality had been given earlier by Iwaniec [15]). Suppose that

$$a_0(k) \le \rho(k) \le a_1(k) \quad \text{if} \quad k \in I; \tag{1.3}$$

$$b_0(m) \le \rho(m) \le b_1(m) \quad \text{if} \quad m \in J; \tag{1.4}$$

then

$$\rho(k)\rho(m) \ge a_0(k)b_1(m) + a_1(k)b_0(m) - a_1(k)b_1(m)$$

for $k \in I$, $m \in J$, and accordingly

$$S \ge S_{0,1} + S_{1,0} - S_{1,1}$$

where

$$S_{i,j} = \sum_{m+k=2n} a_i(k)b_j(m).$$

Here and below, *summations over k and m will always run over I and J respectively.* It now suffices to prove that

$$S_{i,j} = u_i v_j \frac{Y}{\mathcal{L}\mathcal{L}'} \mathfrak{S}(2n)\left(1 + O(\mathcal{L}^{-1})\right)$$

for almost all even $2n$ in $[x, x + x^{\theta_1\theta_2}]$, with $\mathcal{L} = \log x$, $\mathcal{L}' = \log Y$,

$$u_0 > 0.99, \quad u_1 < 1.01, \quad v_0 > 0.05, \quad v_1 < 2.2. \tag{1.5}$$

(The definition of the singular series $\mathfrak{S}(2n)$, which is always positive, is given in §2.) Obviously it is crucial that the constants satisfy

$$u_0 v_1 + u_1 v_0 - u_1 v_1 > 0$$

and this is an easy consequence of (1.5).

The authors of [23], [20], [16], [17], [18] 'sieved J but not I,' in other words, used the simpler inequality

$$\rho(k)\rho(m) \ge \rho(k)b_0(m).$$

Our choice of functions a_0, a_1, b_0, b_1 is based on the sieve method of Harman [7]. In order to make this choice we need to establish classes of sequences $a(k)$ $(k \in I)$, $b(m)$ $(m \in J)$ for which an asymptotic formula

$$\sum_{k+m=2n} a(k)b(m) = uv \frac{Y}{\mathcal{L}\mathcal{L}'} \mathfrak{S}(2n)\left(1 + O(\mathcal{L}^{-1})\right) \tag{1.6}$$

holds for almost all $2n$ in $[x, x + x^{\theta_1\theta_2}]$. We apply the Hardy-Littlewood method, and follow [23] quite closely, to obtain the following result. We write

$$H = Y^{\theta_2}, \quad Q = [H^{\frac{1}{2}}/2], \quad P(z) = \prod_{p<z} p,$$

$$\delta_\chi = \begin{cases} 1 & \text{if } \chi \text{ is the principal character } \chi_0 \pmod q \\ 0 & \text{if } \chi \text{ is non principal } \pmod q. \end{cases}$$

(The letter p is reserved for prime numbers.) By B we denote an absolute constant (not always the same one); ϵ is chosen so that $B\epsilon$ is sufficiently small (whenever necessary).

Theorem 2. *Suppose that the sequences $a(k)$ $(k \in I)$ and $b(m)$ $(m \in J)$ satisfy the following, for every $A > 0$ and $N > C_2(A)$:*
(i) *we have*

$$\sum_{k\in I,\ k\le t} \left(a(k)\chi(k) - \frac{\delta_\chi u}{\mathcal{L}} \right) \ll Y\mathcal{L}^{-A} \tag{1.7}$$

for $t \le x$ and any character $\chi \pmod q$ when $q \le \mathcal{L}^A$;
(ii) *we have*

$$a(k) = O\big(\tau(k)^B\big), \quad b(m) = O\big(\tau(m)^B\big)$$

where τ is the divisor function, and $a(k) = 0$ unless $\big(k, P(\mathcal{L}^A)\big) = 1$;
(iii) *we have*

$$\sum_{m\in J,\ m\le t} \left(b(m)\chi(m) - \frac{\delta_\chi v}{\mathcal{L}'} \right) \ll Y\mathcal{L}^{-A} \tag{1.8}$$

for $t \le 2Y$ and any character $\chi \pmod q$ when $q \le \mathcal{L}^A$;
(iv) *for any $q \le Q$, and any $z \in \left[\frac{qQ}{6Y}, \frac{6qQ}{Y}\right]$, we have*

$$\sum_{\substack{\chi \pmod q \\ \chi\notin E_q}} \int_{Y/2}^{3Y} \left| \sum_{n\in J_y} \beta(m)\chi(m) - \frac{\delta_\chi v}{\mathcal{L}'} \right|^2 dy \ll (qQ)^2 Y\mathcal{L}^{-A}. \tag{1.9}$$

Here $J_y = [y, y + yz]$ and E_q is a set of $O\big(q^{1/2}\mathcal{L}^{-2A}\big)$ characters $\pmod q$. Then (1.6) holds for almost all even integers $2n$ in $[N, N + N^{\theta_1\theta_2}]$.

Remark. Here and later, implied constants depend at most on A.

We shall prove Theorem 2 in §2. In §4, we shall find two sequences $a_0(k)$, $a_1(k)$ which satisfy (i) and (ii), with associated constants u_0, u_1, in the role of u, such that (1.3) and the first two inequalities in (1.5) hold. In §5, we shall find sequences $b_0(m)$, $b_1(m)$ which satisfy (ii), (iii), (iv), with associated constants v_0, v_1 in the role of v, such that (1.4) and the last two inequalities in (1.5) hold. This will establish Theorem 1.

It is worth noting the following consequences of our construction.

Theorem 3. (I) *Let* $x^{0.55+\epsilon} \le M \le x\mathcal{L}^{-1}$. *For all* $q \le \mathcal{L}^A$, $N > C_3(A)$,

$$\frac{0.99M}{\mathcal{L}} < \pi(x;q,a) - \pi(x-M;q,a) < \frac{1.01M}{\mathcal{L}}$$

whenever $(a,q) = 1$.

(II) *Let* $\lambda \ge \frac{1}{16} - 10^{-5}$. *For all integers* $h \le Y$ *with* $O(Y\mathcal{L}^{-A})$ *exceptions,*

$$\pi(h) - \pi(h - h^\lambda) > \frac{0.05h^\lambda}{\log h}.$$

(III) *For* $x > C_4$, *the interval* $[x, x + x^\mu]$ *contains Goldbach numbers. Here* $\mu = 0.0335$.

A result of the form (I) with $q = 1$ is claimed by Lou and Yao [19]. However, there are serious errors in this paper, including the use of the bound

$$\pi(x) - \pi(x-y) \ll \frac{y}{\log x}$$

with y tending to zero. Nevertheless, (I) is not new for $q = 1$, having been found earlier by D.R. Heath-Brown by a method similar to that in his paper [11]. We thank Roger Heath-Brown for making available to us his unpublished notes; our approach is somewhat different.

For results of the type (II), see [21], [7], [8], [33], [16], [17], [18]. The condition on λ in [33] is $\lambda \ge \frac{1}{14} + \epsilon$. Li (work in preparation) replaces $\frac{1}{14}$ by $\frac{1}{15}$. K.C. Wong [34], using a theorem of Watt [32] which would not be helpful in the context of §5, is able to get $\lambda = \frac{1}{18} + \epsilon$. Jia is able to improve this to $\lambda = \frac{1}{20} + \epsilon$ but the details are quite formidable.

To prove Theorem 3 (I), we simply use the fact (already explained above) that (i) of Theorem 2 holds for sequences $a_0(k)$, $a_1(k)$, lying below and above $\rho(k)$, with associated constants in $(0.99, 1.01)$. We then pick out integers congruent to $a \pmod q$ using characters in standard fashion.

To prove Theorem 3 (II), we use the sequence $b_0(k)$ constructed in §6 having the properties

$$\rho(m) \ge b_0(m) \quad \text{when} \quad m \sim Y,$$

$$\int_Y^{(1+\epsilon)Y} \Bigg(\sum_{m \in J_y} \left(b_0(m) - \frac{v}{\mathcal{L}'} \right) \Bigg)^2 dy \ll Q^2 Y \mathcal{L}^{-A}$$

with $v > 0.05$. Here we use property (iv) with $q = 1$ (the set E_1 is obviously empty), and we employ slight variants

$$Q = Y^\lambda, \quad J_y = [\, y,\, y + yQ/Y\,]$$

of earlier definitions. Thus

$$\sum_{y \le k < y + y^\lambda} \rho(k) \ge \sum_{y \le k < y + yQ/Y} b_0(k) > \frac{(v - 2\epsilon) y^\lambda}{L}$$

for all $y \in [\,Y,\, (1+\epsilon)Y\,]$ except for a set of measure $O\big(Y\mathcal{L}^{-A}\big)$. It is now easy to complete the proof of Theorem 3 (II).

To prove Theorem 3 (III) we argue as in Montgomery and Vaughan [22], employing Theorem 3 (II) in conjunction with the lower bound

$$\pi(x) - \pi(x - y) \gg \frac{y}{\log x}$$

for $x^{0.535} \le y \le x$ (Baker and Harman [2]). The constant 0.0335 is larger than 0.535/16.

2. Proof of Theorem 2

We write $e(\theta) = e^{2\pi i \theta}$,

$$S_1(\alpha) = \sum_k a(k) e(k\alpha), \qquad S_2(\alpha) = \sum_m b(m) e(m\alpha).$$

Thus

$$\sum_{k+m=2n} a(k) b(m) = \int_{1/Q}^{1+1/Q} S_1(\alpha) S_2(\alpha) e(-2n\alpha)\, d\alpha.$$

We divide up the interval $[\,1/Q,\, 1 + 1/Q\,]$ into Farey arcs of order Q, writing $I_{q,r}$ for the arc with centre at r/q. Thus

$$I_{q,r} \subset \left[\frac{r}{q} - \frac{1}{qQ},\, \frac{r}{q} + \frac{1}{qQ} \right]$$

for $q \le Q$, $1 \le r \le q$, $(r,q) = 1$. Let

$$I'_{q,r} = \left[\frac{r}{q} - \frac{\mathcal{L}^{4A}}{qY},\, \frac{r}{q} + \frac{\mathcal{L}^{4A}}{qY} \right], \qquad I''_{q,r} = \begin{cases} I_{q,r} \backslash I'_{q,r} & \text{if} \quad q \le \mathcal{L}^{2A} \\ I_{q,r} & \text{if} \quad q > \mathcal{L}^{2A}. \end{cases}$$

The major and minor arcs are defined by

$$\mathfrak{M} = \bigcup_{q \le \mathcal{L}^{2A}} \bigcup_{r=1}^{q}{}^{*} I'_{q,r}, \qquad \mathfrak{m} = [\,1/Q,\, 1 + 1/Q\,] \backslash \mathfrak{M}$$

respectively. As usual, an asterisk denotes a restriction to those r coprime to q.

Let us write

$$\mathfrak{S}(2n) = 2\prod_p\left(1-\frac{1}{(p-1)^2}\right)\prod_{\substack{p|n\\p>2}}\left(\frac{p-1}{p-2}\right),$$

$$c_q(m) = \sum_{r=1}^{q}{}^{*} e\left(\frac{mr}{q}\right), \qquad \tau(\chi) = \sum_{r=1}^{q}{}^{*}\chi(r)e\left(\frac{r}{q}\right).$$

For the well-known formula

$$\mathfrak{S}(2n) = \sum_{q=1}^{\infty}\frac{\mu^2(q)}{\phi^2(q)}c_q(-2n)$$

see [31], (3.24). We recall the well-known results ([5], pages 66, 67 for example) for the Gauss sum :

$$|\tau(\chi)| \le q^{\frac12}, \quad \tau(\chi_0) = \mu(q). \tag{2.1}$$

To prove Theorem 2 it suffices to show that

$$\sum_{2n\in K}\left|\int_{1/Q}^{1+1/Q} S_1(\alpha)S_2(\alpha)e(-2n\alpha)\,d\alpha - \frac{uvY}{\mathcal{L}\mathcal{L}'}\mathfrak{S}(2n)\right|^2 \ll HY^2\mathcal{L}^{-A-7}.$$

We accomplish this in two stages by proving

$$\sum_{2n\in K}\left|\int_{\mathfrak{M}} S_1(\alpha)S_2(\alpha)e(-2n\alpha)\,d\alpha - \frac{uvY}{\mathcal{L}\mathcal{L}'}\mathfrak{S}(2n)\right|^2 \ll HY^2\mathcal{L}^{-A-7}, \tag{2.2}$$

$$\sum_{2n\in K}\left|\int_{\mathfrak{m}} S_1(\alpha)S_2(\alpha)e(-2n\alpha)\,d\alpha\right|^2 \ll HY^2\mathcal{L}^{-A-7}. \tag{2.3}$$

We begin the proof of (2.2) by replacing the S_j by suitable approximations. For $q \le \mathcal{L}^{2A}$ we have $(k,q)=1$ whenever $a(k)\ne 0$. Hypothesis (i) suggests the use of the identity

$$\begin{aligned} S_1\left(\frac{r}{q}+\eta\right) &= \sum_k a(k)e(k\eta)\frac{1}{\phi(q)}\sum_\chi \tau(\bar\chi)\chi(kr)\\ &= \frac{1}{\phi(q)}\sum_\chi \tau(\bar\chi)\chi(r)\sum_k a(k)\chi(k)e(k\eta)\\ &= S_1'\left(\frac{r}{q}+\eta\right) + \sum_\chi\frac{\tau(\bar\chi)}{\phi(q)}\chi(r)\sum_k\left(\left(a(k)\chi(k)-\frac{u\delta_\chi}{\mathcal{L}}\right)e(k\eta)\right). \end{aligned} \tag{2.4}$$

Here, for $r/q+\eta \in I'_{q,r}$,

$$S_1'\left(\frac{r}{q}+\eta\right) = \frac{\mu(q)}{\phi(q)}\frac{u}{\mathcal{L}}\sum_k e(k\eta).$$

(We have used (2.1) to rewrite the contribution from χ_0.)

To bound the double sum in (2.4) we use partial summation. With

$$v(t) = \sum_{\substack{k\in I\\ k<t}}\left(a(k)\chi(k) - \frac{u\delta_\chi}{\mathcal{L}}\right),$$

hypothesis (i) yields

$$\begin{aligned}\sum_k\left(a(k)\chi(k)-\frac{u\delta_\chi}{\mathcal{L}}\right)e(k\eta) &= \int_I e(t\eta)\,dv(t)\\ &= \left[e(t\eta)v(t)\right]_{x-2Y}^{x} - 2\pi i\eta\int_I e(t\eta)v(t)dt\\ &\ll \left(1+Y|\eta|\right)\max_{t\in I}|v(t)| \ll \mathcal{L}^{4A}Y\mathcal{L}^{-13A} \ll Y\mathcal{L}^{-9A}.\end{aligned}$$

Here and below we suppose that $x > C_2(A')$ where A' is sufficiently large in terms of A. The double sum in (2.4) is thus $\ll Y\mathcal{L}^{-8A}$.

Since $\mathfrak{M}$ has Lebesgue measure $\ll \mathcal{L}^{6A}Y^{-1}$,

$$\int_{\mathfrak{M}}\left|(S_1-S_1')S_2\right|d\alpha \ll \mathcal{L}^{6A}Y^{-1}.Y\mathcal{L}^{-8A}.Y\mathcal{L}^{B} \ll Y\mathcal{L}^{-A}. \qquad (2.5)$$

(We have used hypothesis (ii) to get $|S_2| \ll \sum_m |b(m)| \ll Y\mathcal{L}^B$.)

The bound (2.5) clearly permits us to replace S_1 by S_1' in proving (2.2). In exactly the same way, we may replace $S_1'S_2$ by $S_1'S_2'$, where

$$S_2'\left(\frac{r}{q}+\eta\right) = \frac{\mu(q)}{\phi(q)}\frac{v}{\mathcal{L}'}\sum_m e(m\eta) \quad \text{for} \quad \frac{r}{q}+\eta\in I'_{q,r};$$

that is, we need only prove the analogue of (2.2) for $S_1'S_2'$.

We may rewrite the integral

$$\int_{\mathfrak{M}} S_1'(\alpha)S_2'(\alpha)e(-2n\alpha)\,d\alpha$$

in the form

$$\frac{uv}{\mathcal{L}\mathcal{L}'}\sum_{q\le\mathcal{L}^{2A}}\sideset{}{^*}\sum_{r=1}^{q}\frac{\mu^2(q)}{\phi^2(q)}e\left(-\frac{2nr}{q}\right)\int_{-\eta_0}^{\eta_0}\sum_k\sum_m e((k+m-2n)\eta)\,d\eta, \qquad (2.6)$$

where $\eta_0 = \mathcal{L}^{4A}/qY$. If we replace the domain of integration in (2.6) by $[-\frac{1}{2}, \frac{1}{2}]$, we introduce an error

$$\ll \sum_{q \le \mathcal{L}^{2A}} \frac{1}{\phi(q)} \int_{\eta_0}^{\frac{1}{2}} \frac{d\eta}{\eta^2} \ll \sum_{q \le \mathcal{L}^{2A}} \frac{qY}{\phi(q)\mathcal{L}^{4A}} \ll Y\mathcal{L}^{-A}.$$

This replaces the left-hand side of (2.6) by

$$\frac{uv}{\mathcal{L}\mathcal{L}'} \sum_{q \le \mathcal{L}^{2A}} \frac{\mu^2(q)}{\phi^2(q)} c_q(-2n) \sum_{\substack{k \\ k+m=2n}} \sum_{m} 1 = \frac{uv}{\mathcal{L}\mathcal{L}'} \sum_{q \le \mathcal{L}^{2A}} \frac{\mu^2(q)}{\phi^2(q)} c_q(-2n)(Y + O(1))$$

for $2n \in K$. Consequently,

$$\begin{aligned} \sum_{2n \in K} \left| \int_{\mathfrak{M}} S_1'(\alpha) S_2'(\alpha) e(-2n\alpha) d\alpha - \frac{uvY}{L^2} \mathfrak{S}(2n) \right|^2 & \\ \ll HY^2\mathcal{L}^{-A-7} + \sum_{2n \in K} Y^2 \left| \sum_{q > \mathcal{L}^{2A}} \frac{\mu^2(q)}{\phi^2(q)} c_q(-2n) \right|^2 & \\ \ll HY^2\mathcal{L}^{-A-7} + Y^2 \sum_{2n \in K} \left\{ \sum_{d \mid 2n} \frac{1}{\phi(d)} \min\left(\frac{d}{\mathcal{L}^{2A}}, 1 \right) \right\}^2 . & \qquad (2.7) \end{aligned}$$

For the last step we use Vaughan [31, (3.23)]. Now

$$\sum_{2n \in K} \left\{ \sum_{\substack{d \mid 2n \\ d \le \mathcal{L}^{2A}}} \frac{1}{\phi(d)} \frac{d}{\mathcal{L}^{2A}} \right\}^2 \ll \mathcal{L}^{-4A+1} \sum_{j \in K} \tau^2(j) \ll H\mathcal{L}^{-A-7}, \qquad (2.8)$$

$$\sum_{2n \in K} \left\{ \sum_{\substack{d \mid 2n \\ d > \mathcal{L}^{2A}}} \frac{1}{\phi(d)} \right\}^2 \ll \mathcal{L}^{-4A+1} \sum_{j \in K} \tau^2(j) \ll H\mathcal{L}^{-A-7}. \qquad (2.9)$$

The analogue of (2.2) for $S_1' S_2'$ now follows from (2.7)–(2.9). This completes the proof of (2.2).

We now turn to the sum in (2.3), which may be rewritten as

$$\begin{aligned} \sum_{2n \in K} \int_{\mathfrak{m}} S_1(\zeta) S_2(\zeta) e(-2n\zeta) \int_{\mathfrak{m}} \overline{S_1(\alpha)}\, \overline{S_2(\alpha)}\, e(2n\alpha)\, d\alpha\, d\zeta & \\ \ll \int_{\mathfrak{m}} |S_1(\zeta) S_2(\zeta)| \int_{\mathfrak{m}} |S_1(\alpha) S_2(\alpha)| \min\left(H, \frac{1}{\|2(\alpha - \zeta)\|} \right) d\alpha\, d\zeta & \\ \ll (Y\mathcal{L}^B)^{\frac{3}{2}} \sup_{\zeta \in [0,1]} \left(\int_{m} |S_2(\alpha)|^2 \min\left(H, \frac{1}{\|2(\alpha - \zeta)\|} \right)^2 d\alpha \right)^{\frac{1}{2}} . & \qquad (2.10) \end{aligned}$$

For the last step we have used the Cauchy-Schwarz inequality, the bound

$$\int_0^1 \left|S_1(\alpha)\right|^2 d\alpha = \sum_k \left|a(k)\right|^2 \ll \sum_k \tau(k)^{2B} \ll Y\mathcal{L}^B,$$

and the corresponding bound for S_2.

In view of (2.10), it suffices to show that

$$\sup_\zeta \int_{\mathcal{I}(\zeta)} \left|S_2(\alpha)\right|^2 d\alpha \ll Y\mathcal{L}^{-3A}, \tag{2.11}$$

where $\mathcal{I}(\zeta) = (\zeta - 1/H,\ \zeta + 1/H) \cap \mathfrak{m}$. Since $I_{q,r}$ has length at least

$$1/Q^2 > 2/H,$$

there are at most two punctured arcs $I''_{q,r}$ with $q \leq Q$ and $(r, q) = 1$, which intersect $(\zeta - 1/H,\ \zeta + 1/H)$. Instead of (2.11), then, we may show that

$$\int_{I''_{q,r}} \left|S_2(\alpha)\right|^2 d\alpha \ll Y\mathcal{L}^{-3A}$$

for $q \leq Q,\ (r, q) = 1$.

By the analogue of (2.4) for S_2, it suffices to show that

$$\int_{\eta + r/q \in I''_{q,r}} \left|S_2'\left(\frac{r}{q} + \eta\right)\right|^2 d\eta \ll Y\mathcal{L}^{-3A}, \tag{2.12}$$

$$\frac{q}{\phi^2(q)} \int_{-1/qQ}^{1/qQ} \left\{\sum_\chi \left|W(\chi, \eta)\right|\right\}^2 d\eta \ll Y\mathcal{L}^{-3A}. \tag{2.13}$$

Here

$$W(\chi, \eta) = \sum_m \left(b(m)\chi(m) - \frac{v\delta_\chi}{\mathcal{L}'}\right) e(m\eta).$$

The left hand side of (2.12) is

$$\ll \frac{1}{\phi^2(q)} \frac{1}{\mathcal{L}^2} \int_{I(\eta)} \min\left(\frac{1}{Y^2}, \frac{1}{\eta^2}\right) d\eta \tag{2.14}$$

with

$$I(\eta) = \begin{cases} \left[\mathcal{L}^{4A}/qY,\ \frac{1}{2}\right] & \text{if} \quad q \leq \mathcal{L}^{2A} \\ \left[0,\ \frac{1}{2}\right] & \text{if} \quad q > \mathcal{L}^{2A}. \end{cases}$$

For $q \leq \mathcal{L}^{2A}$, the expression in (2.14) is $\ll q^{-2}qY\mathcal{L}^{-4A} \ll Y\mathcal{L}^{-3A}$; for $q > \mathcal{L}^{2A}$, we have instead the bound $\ll q^{-2}Y \ll Y\mathcal{L}^{-3A}$. This establishes (2.12).

We split the sum over χ into sums over E_q and

$$E_q^c = \{\chi \ (\bmod q) \colon \chi \notin E_q\}.$$

Applying Cauchy's inequality to each subsum, it suffices to prove that

$$\frac{q|E|}{\phi^2(q)} \int_{-1/qQ}^{1/qQ} \sum_{\chi \in E} \left|W(\chi,\eta)\right|^2 d\eta \ll Y\mathcal{L}^{-3A} \tag{2.15}$$

for $E = E_q$, E_q^c in order to establish (2.12).

By hypothesis (ii), (iv) and Parseval's inequality the left hand side of (2.15) is bounded when $E = E_q$ by

$$\ll \frac{q}{\phi^2(q)}|E_q| \int_{-\frac12}^{\frac12} \sum_{\chi \in E_q} \left|W(\chi,\eta)\right|^2 d\eta \ll \frac{q}{\phi^2(q)}|E_q|^2 \sum_m \left(\left|b(m)\right|^2 + 1\right)$$
$$\ll q^{-1}\mathcal{L}|E_q|^2 Y\mathcal{L}^B \ll Y\mathcal{L}^{-3A}.$$

For $E = E_q^c$ we appeal to Gallagher's Lemma ([21], Lemma 1.9):

$$\int_{-1/qQ}^{1/qQ} \left|W(\chi,\eta)\right|^2 d\eta \ll \frac{1}{(qQ)^2} \int_{Y/2}^{3Y} \left\{ \sum_{m \in J'_y} \left(b(m)\chi(m) - \frac{v\delta_\chi}{\mathcal{L}}\right)\right\}^2 dy \tag{2.16}$$

where

$$J'_y = \left[y - \tfrac12 qQ,\, y + \tfrac12 qQ\right].$$

Recalling that $J_y = [y,\, y + yz]$, for some $z \in [qQ/6Y,\, 6qQ/Y]$, the right hand side of (2.16) is

$$\ll U_\chi := \frac{1}{(qQ)^2} \int_{Y/2}^{3Y} \left\{ \sum_{m \in J_y} \left(b(m)\chi(m) - \frac{v\delta_\chi}{\mathcal{L}}\right)\right\}^2 dy;$$

this may be demonstrated using a device of Saffari and Vaughan ([29], proof of (6.21)). By hypothesis (iv),

$$\sum_{\chi \in E_q^c} U_\chi \ll Y\mathcal{L}^{-4A}.$$

The bound (2.15) with $E = E_q^c$ follows at once. This completes the proof of Theorem 2.