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Harmonic Maps, Loop Groups, and Integrable Systems

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Preface

Motivation: Harmonic maps

The principal motivation for this book was provided by certain recent advances in the theory of harmonic maps, which depend on ideas from the theory of integrable systems.

The concept of harmonic map is a generalization of the concept of geodesic. The harmonic maps in this book go from Riemann surfaces to compact Lie groups or symmetric spaces; they are, therefore, two-dimensional analogues of geodesics. They encompass many fundamental examples in differential geometry, such as minimal surfaces, which have been studied by geometers over a long period of time. Nevertheless, important new discoveries are still being made, and major open problems remain. Since the late 1970s, the field has acquired new vitality from mathematical physics, in the guise of the non-linear sigma model or chiral model. As a result, harmonic maps have attracted the attention of a much wider audience than before, both within the mathematical community and beyond.

One of the themes of research in this area during the last 25 years or so is the "classification" of such harmonic maps, i.e., the description (or parametrization) of harmonic maps from Riemann surfaces to compact Lie groups or symmetric spaces, in terms of well known maps. There are several reasons for doing this. The most obvious one is that such a description provides the "general solution" of the relevant harmonic map equation. Another is that such a description should be useful in describing the "moduli space" (or, more accurately, the parameter space) of solutions of the harmonic map equation. The nature of this problem turns out to be algebraic, rather than analytical; it is global rather than local. The methods used to study the problem are therefore closer to algebra and topology than to analysis. There are of course other problems of interest, and progress has been made with these as well. But this book is concerned entirely with the classification problem. The survey articles of Eells and Lemaire [1978; 1988] may be consulted for a modern view of the history of harmonic maps and its problems, including the progress made on the classification problem before 1988.

A turning point in the theory was the idea that the harmonic map equation is a kind of "integrable system". This idea first arose explicitly in the mathematical physics literature, for example in Pohlmeyer [1976]; Zakharov and Mikhailov [1978]; Zakharov and Shabat [1979]. The harmonic map equation was reformulated as a kind of Lax equation "with parameter". This was taken up in Uhlenbeck [1989], where the first new results were obtained. At approximately the same time, great progress was being made with integrable systems such as the KdV equation by employing certain infinite dimensional Lie algebras (affine Kac-Moody Lie algebras) and Lie groups (loop groups). The potential of these Lie theoretic methods in the harmonic map problem was suggested in Segal [1989], where another proof of Uhlenbeck's results was given.

Further developments came rather slowly, reflecting the fact that the harmonic map problem is quite different from those problems like the KdV equation which had been tackled so successfully. Lax equations and loop groups provided a solution of the classification problem for harmonic maps from S^2 to the unitary group U_n (in Uhlenbeck [1989]; Valli [1988]; Wood [1989]; Segal [1989]), where "solution" is taken to mean "reduction to holomorphic data". However, the question of generalization – to higher genus Riemann surfaces, or other Lie groups and symmetric spaces – still seemed inaccessible, and in any case the solution for U_n did not amount to giving explicit formulae for harmonic maps. Thus, the "integrable systems approach" seemed at this point not to provide the breakthrough that had been hoped for.

A breakthrough came in the genus one case, i.e., harmonic maps from a torus. Wente's counterexample to the Hopf conjecture (Wente [1986]) was shown to arise from the integrable systems point of view in Pinkall and Sterling [1989]. This was generalized significantly by various authors (for example, in Burstall *et al.* [1993]), thus providing further evidence of the usefulness of integrable systems and loop groups. Meanwhile, the loop group approach to the genus zero case (i.e., harmonic maps from S^2) was pursued in Bergvelt and Guest [1991]; Guest and Ohnita [1993]. This yielded new results on the connected components and fundamental group of the space of harmonic maps, in which the role of the "symmetry group" is essential.

With the benefit of hindsight, and a new point of view suggested in Dorfmeister *et al.* [to appear], the genus zero case turns out to have some aspects in common with the genus one case. This leads to new explicit formulae in the genus zero case, and also a way of generalizing the results of Uhlenbeck and Segal. It seems reasonable to conclude that the basis of a unifying theory has now been been established. From a naive computational point of view, the basic phenomenon is that the harmonic maps considered here may be expressed in terms of "factorization of exponentials".

This book is based on lectures in which I attempted to present this unifying theory in a straightforward manner. I certainly failed in this attempt, as the time available in each series of lectures permitted only about a third of the material to be covered. In the present more comprehensive exposition, however, I have tried to maintain some of the informal style of the lectures. Wherever possible, generality and completeness have been sacrificed in favour of accessibility.

Integrable systems, loop groups, and harmonic maps each have their own well developed literature, and it was not my intention merely to duplicate this. However, I hope that the modest introductions to the theories of integrable systems and loop groups given here may be useful, even for the reader whose main interest is not in harmonic maps.

A summary of the topics will be given shortly. For the benefit of the knowledgeable reader, Chapter 26 contains a more technical summary, together with information on topics which are not treated in this book.

What is an integrable system?

Whoever wishes to learn something about "integrable systems" faces at least two difficulties. One is the breadth of the subject: It ranges over mechanics, differential equations, global analysis, algebraic geometry, and Lie theory; and these are just some of the mathematical aspects, ignoring the vast intersection with physics. Moreover – this is the second difficulty – the subject accommodates a whole range of points of view, from very "pure" to very "applied". Some authors emphasize general concepts such as symplectic or Poisson structures. They naturally focus attention on those examples which best illustrate the particular general theory under discussion. Other authors give priority to the examples themselves – classical mechanical systems, the Toda lattice, the sine-Gordon equation, or the KdV equation perhaps – referring to the general theories only in passing. Needless to say, this leads to quite different styles of exposition, which can be disconcerting for the beginner.

There is one common thread in all this, namely the idea of a symmetry group of a differential equation. The nicest and most natural equations (such as those which occur in the "real world") often admit symmetry groups. The existence of a large enough symmetry group leads to the possibility of solving the differential equation by *algebraic* means, and this is perhaps the fundamental property of an integrable system.

The term "integrable system" is used rather loosely in the literature, and a precise definition will not be given in this book. But this is not really a practical disadvantage; after all, there is no precise definition of an elephant, yet in practice one may recognize an elephant by various fundamental properties. Similarly, one has various fundamental properties of an integrable system: It is a differential equation, usually with geometrical or physical significance, admitting a large symmetry group, which is solvable (integrable) by algebraic means, etc. As another substitute for a precise definition, one could exhibit a typical example, such as the Toda lattice. Whatever an integrable system is, the Toda lattice certainly is one! For this reason, the Toda lattice plays a prominent role in this book. Another reason for choosing the Toda lattice here, however, is that it happens to be *directly* related to (certain kinds of) harmonic maps.

This book deals only with that part of the theory of integrable systems which is immediately applicable to the classification of harmonic maps from Riemann surfaces to Lie groups or symmetric spaces. Even this is not treated comprehensively here; to do so would have stretched the book (and its author) too far. The biggest omission is the theory of spectral curves, and indeed the entire algebraic geometry point of view. Another major gap is the theory of harmonic maps from Lorentzian (rather than Riemannian) surfaces, which is closely related to soliton theory.

As far as the general theory of integrable systems is concerned, this book is even less ambitious. Nevertheless, the topics which arise here – such as Lax equations, zero-curvature equations, hidden symmetry groups, Bruhat decompositions, τ-functions, Riemann-Hilbert problems, dressing transformations – do illustrate some important features of the theory.

Brief description of topics

Part I One-dimensional integrable systems.

The first two chapters give a very brief introduction to Lie groups and Lie algebras, concentrating on the exponential map and the adjoint representation, in the context of matrix groups. Chapter 3 introduces a fundamental technical tool which will be used repeatedly: the Iwasawa decomposition of a Lie group.

Chapter 4 discusses an important classical source of motivation, Hamilton's equations. The concept of a general Hamiltonian system is illustrated by a famous example: a "height function" on an adjoint orbit of a compact Lie group. The Hamiltonian function, its corresponding Hamiltonian vector field, and the associated one parameter group of diffeomorphisms are easily described and explicitly computed for this example.

Chapters 5 to 8 constitute a survey of the (one-dimensional, open) Toda lattice, a "model" integrable system. This is simply a first order ordinary differential equation, but with a very interesting algebraic structure. The Lax form of the equation is given in Chapter 5, from which the general solution is obtained explicitly. The formula for this solution involves exponentiating a (linear) matrix function, then performing a certain matrix factorization.

In Chapter 6, the story is repeated, but this time entirely in Lie algebraic terms. This paves the way for a discussion of the generalized Toda lattice (for a general Lie algebra, and for general invariant Hamiltonian functions) in Chapter 7. The role of the classical Toda lattice as just one member of a hierarchy of integrable systems is thus revealed. Chapter 8 discusses further generalizations, such as the concept of an R-matrix, and a curious relationship between the Toda lattice and the example of Chapter 4.

Part II Two-dimensional integrable systems.

Chapters 9 to 22 are intended to be parallel to Chapters 1 to 8, but using infinite dimensional Lie groups instead of finite dimensional Lie groups, zero-curvature equations instead of Lax equations, and the two-dimensional Toda lattice (and harmonic map equation) instead of the one-dimensional Toda lattice.

The first two chapters provide motivation. Zero-curvature equations are treated as two-dimensional analogues of Lax equations, and our two main examples are introduced (the periodic two-dimensional Toda lattice, and the harmonic map equation for maps from surfaces to Lie groups). There is now little hope of writing down "the general solution". As a substitute, one has the method of dressing transformations, which converts a trivial solution into a non-trivial solution, by applying an element of an infinite dimensional Lie group. It is at this point that loop groups appear.

Chapter 11 gives the bare facts concerning loop groups and loop algebras which will be needed. Chapter 12 gives an equally brief description of the Iwasawa decomposition for loop groups.

The two-dimensional Toda lattice is the subject of Chapters 13 to 15. In Chapter 13 the equation is formulated precisely in loop theoretic terms. In Chapters 14 and 15 a family of explicit solutions is obtained by applying dressing transformations to a trivial solution. (Chapter 14 is devoted to an elementary explanation of τ-functions, primarily for the one-dimensional case, as preparation for Chapter 15.)

Harmonic maps from surfaces to compact Lie groups and symmetric spaces are the subject of Chapters 16 to 22. Chapter 16 gives the precise loop theoretic formulation for harmonic maps into Lie groups, and Chapter 18 does the same for symmetric spaces. In each case a suitable symmetry group is identified, which acts on solutions (i.e., harmonic maps) by dressing transformations. Chapters 17 (for Lie groups) and 19 (for symmetric spaces) describe some well known results which are available when the domain surface is S^2, or more generally when the harmonic maps have "finite uniton number".

In Chapter 20 a geometrical method for studying harmonic maps is introduced, based on the Bruhat decomposition of the Grassmannian model of a loop group. As illustrations, a new proof is given of the classification of harmonic maps from S^2 into complex projective space, together with some new estimates for the "uniton number" of harmonic maps from S^2 into complex Grassmannians.

Chapter 21 discusses primitive maps and their relevance to harmonic maps, and a relationship between the two-dimensional Toda lattice and harmonic maps.

Using the method of Chapter 20, it is shown in Chapter 22 how the harmonic maps obtained so far are given by formulae which are surprisingly analogous to the "factorization of exponentials" formulae appearing as the solutions to the one-dimensional Toda lattice. This sets the scene for Part III.

Part III One-dimensional and two-dimensional integrable systems.

Chapters 23 to 25 bring together Parts I and II by showing that they are not just analogous, but in fact very directly related. This direct relationship is based on the construction (described in Chapter 23) of a solution of a zero-curvature equation from the solutions of two Lax equations, the "$1+1=2$" principle.

In Chapter 24 this is applied to the harmonic map equation; the solutions obtained by this method are called harmonic maps of "finite type". Like all our previous examples, they are given by "factorization of exponentials" formulae. More generally, one has primitive maps of finite type.

Harmonic maps of finite type are quite different from harmonic maps of finite uniton number, and their study is only just beginning. In Chapter 25 the first important example is described, namely harmonic maps of finite type from a torus to S^2.

Background knowledge required

Only the following knowledge will be taken for granted: linear algebra, elementary definitions of topology, basic theory of ordinary differential equations, and elementary properties of differentiable manifolds (such as the concepts of tangent bundle and vector field).

I have tried not to assume too much in the way of Lie theory, or the theory of loop groups, or for that matter the theory of harmonic maps; but the reader who has some knowledge of at least one of these areas will find the material much easier to read.

There exist beautiful generalizations of many aspects of the theory presented here, but I have resisted the temptation to pursue them. For example, most of the time I consider only matrix groups, primarily the orthogonal and unitary groups, rather than general Lie groups. I have left symplectic and Poisson geometry in the background, where they certainly belong in an introductory course.

The exercises are quite numerous in Part I, where it is hoped that they will be more useful. They die out rapidly thereafter. On the other hand, proofs become progressively more detailed. It is an attractive feature of the topics discussed here that the proofs are usually quite elementary and devoid of mysterious technicalities. In the initial chapters, however, most proofs are of standard results, and so they are sketched very briefly or omitted.

The references include most of the recent literature on *connections* between harmonic maps, loop groups, and integrable systems. But within each of these three areas, the references are not at all comprehensive. In many cases I have referred to books rather than original articles; I found Perelomov [1990] and Fordy and Wood [1994] particularly helpful. Other important books that should be mentioned are Carter *et al.* [1995] (for an accessible introduction to Lie theory); Kac [1990]; Pressley and Segal [1986] (for infinite dimensional Lie algebras and Lie groups); Urakawa [1993] (for harmonic maps); and Arnold [1978]; Arnold and Novikov [1990; 1994]; Faddeev and Takhtajan [1987]; Fomenko and Trofimov [1988]; Gu [1995]; Guillemin and Sternberg [1984]; Newell [1985]; Novikov *et al.* [1984] (for integrable systems). The bibliographical comments at the ends of appropriate chapters provide some pointers to the original sources. I apologize for the inevitable omissions here, particularly of the Russian and Japanese literature.

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