Part I

One-dimensional integrable systems

Chapter 1: Lie Groups

I. Definitions and examples.

We assume that the reader is familiar with the idea of a (smooth) manifold. For our purposes, it will be enough to think of a manifold which is embedded (as a submanifold) in some $\mathbf{R}^n$. For example, the sphere $S^{n-1} = \{x \in \mathbf{R}^n \mid ||x|| = 1\}$ is embedded in $\mathbf{R}^n$.

We assume also that the reader understands the concept of tangent bundle. If X is a submanifold of $\mathbf{R}^n$, the tangent space to X at a point $x \in X$ is the space of "tangent vectors to curves in X through x", i.e.,

$$T_x X = \{\gamma'(0) \mid \gamma : (-\epsilon, \epsilon) \to \mathbf{R}^n, \gamma(-\epsilon, \epsilon) \subseteq X, \gamma(0) = x\}.$$

Here, $\gamma'(0) = \frac{d}{dt}\gamma(t)|_0 = D\gamma_0(\frac{d}{dt})$. (In the case of S^{n-1}, it is easy to verify the usual description of $T_x S^{n-1}$ as the set of all vectors which are orthogonal to x.) The tangent bundle of X can then be defined as

$$TX = \{(x, U) \in X \times \mathbf{R}^n \mid U \in T_x X\}.$$

This is a submanifold of $X \times \mathbf{R}^n$, and there is a natural projection map $\pi : TX \to X$. If $f : X \to Y$ is a (smooth) map, then the derivative of f is a map $Df : TX \to TY$.

Using the tangent bundle, one can define another basic object: A vector field on X is a (smooth) map $V : X \to TX$ such that $\pi \circ V$ is the identity map of X. The value of V at x will be denoted by V_x; V_x is an element of $T_x X$. An important property of vector fields is that they act on (smooth) functions "by differentiation": If V is a vector field, and $f : X \to \mathbf{R}$, then we obtain a function Vf by means of the formula $(Vf)(x) = (Df)_x(V_x)$. Here we use the standard convention that $T_u\mathbf{R}$ is identified canonically with $\mathbf{R}$, for any $u \in \mathbf{R}$.

Definition. *Let X be a (smooth) manifold. We say that X is a (real) Lie group if*

(1) X has a group structure, $\circ$, and

(2) the map $X \times X \to X$, $(x, y) \mapsto x \circ y^{-1}$ is smooth.

A complex Lie group may be defined in a similar way: It is a complex manifold X, with a group structure, such that the map $X \times X \to X$, $(x, y) \mapsto x \circ y^{-1}$ is complex analytic (holomorphic).

The main examples of Lie groups are *matrix groups.*

Examples:

(1) $M_n\mathbf{R}$ = {real $n \times n$ matrices}, $\circ$ = matrix addition. Similarly for $M_n\mathbf{C}, M_n\mathbf{H}$. More generally, any real (or complex) vector space is a real (or complex) Lie group.

(2) $GL_n\mathbf{R} = \{A \in M_n\mathbf{R} \mid A \text{ is invertible}\}$, $\circ$ = matrix multiplication. Similarly for $GL_n\mathbf{C}$.

(3) $SL_n\mathbf{R} = \{A \in GL_n\mathbf{R} \mid \det A = 1\}$, $\circ$ = matrix multiplication. Similarly for $SL_n\mathbf{C}$.

(4) $O_n = \{A \in M_n\mathbf{R} \mid A^t = A^{-1}\}$, $\circ$ = matrix multiplication. Similarly we have $U_n = \{A \in M_n\mathbf{C} \mid A^* = A^{-1}\}$, $Sp_n = \{A \in M_n\mathbf{H} \mid A^* = A^{-1}\}$.

(5) $SO_n = \{A \in O_n \mid \det A = 1\}$, $\circ$ = matrix multiplication. Similarly for SU_n.

Exercises:

(1.1) In the above list of examples, which are real Lie groups? Which are complex Lie groups?

(1.2) Why are $GL_n\mathbf{H}, SL_n\mathbf{H}$ and SSp_n omitted from the above list of examples?

(1.3) Show that $GL_n\mathbf{C}$ is connected.

(1.4) Show that SO_n is compact and connected.

(1.5) Show that O_n has two connected components, each of which is diffeomorphic to SO_n. Is it true that O_n is isomorphic (as a group) to $SO_n \times \{\pm I\}$?

Definition. *Let G_1, G_2 be Lie groups. Let $\Theta : G_1 \to G_2$ be a (smooth) map. We say that Θ is a (Lie group) homomorphism if $\Theta(gh) = \Theta(g)\Theta(h)$ for all $g, h \in G$.*

Similarly, we define the concepts of monomorphism, epimorphism, isomorphism, etc.

Example:

The determinant map $\det : GL_n\mathbf{R} \to \mathbf{R}^*$ is a homomorphism. (Here, $\mathbf{R}^* = GL_1\mathbf{R}$, the group of non-zero real numbers.)

The concept of "subgroup" requires a little care:

Definition. *Let G_1, G_2 be Lie groups, such that G_1 is an (algebraic) subgroup of G_2. We say that G_1 is a Lie subgroup of G_2 if the inclusion map $G_1 \to G_2$ is an embedding.*

If G_1 is an (algebraic) subgroup of G_2, and also a submanifold, then G_1

is certainly a Lie subgroup of G_2. However, for reasons which will become clear in the next chapter, we do not *insist* that a Lie subgroup should also be a submanifold. The standard example of this is given by the (algebraic) subgroup $G_1 = \{[ta, tb] \mid t \in \mathbf{R}\}$ of the Lie group $G_2 = \mathbf{R}^2/\mathbf{Z}^2$ (for a fixed choice of $a, b \in \mathbf{R}$ with $(a, b) \neq (0, 0)$). We give G_1 the structure of a Lie group by using the natural homomorphism $\mathbf{R} \to G_1$, $t \mapsto [ta, tb]$. There are two cases to consider: (1) G_1 is isomorphic (as a Lie group) to $\mathbf{R}/\mathbf{Z}$ if $b = 0$ or if a/b is rational; (2) G_1 is isomorphic to $\mathbf{R}$ if a/b is irrational. In both cases, G_1 is a Lie subgroup of G_2. But only in case (1) is G_1 a submanifold of G_2.

It is well known that a Lie subgroup is a submanifold if and only if it is closed (see Varadarajan [1984], Theorem 2.5.4). From Chapter 3 onwards, we shall usually abbreviate the expression "closed Lie subgroup" to "subgroup".

II. The exponential map.

Let G be a Lie group. We use the following standard notation:

$e =$ the identity element of G ($= I$ if G is a matrix group)

$\mathbf{g} = T_e G =$ the tangent space to G at e.

The relationship between G and $\mathbf{g}$ is very important. It is useful, therefore, to have an explicit description of $\mathbf{g}$. Here are some examples, for matrix groups:

Examples:

(1) $T_e M_n \mathbf{R} = M_n \mathbf{R}$ (because $M_n \mathbf{R}$ is a vector space).

(2) $T_e GL_n \mathbf{R} = M_n \mathbf{R}$ (because $GL_n \mathbf{R}$ is an open subset of a vector space).

(3) $T_e O_n = \text{skew}_n \mathbf{R} = \{A \in M_n \mathbf{R} \mid A^t = -A\}$.

(Proof: If $X \subseteq \mathbf{R}^n$, we use the description of $T_x X$ given earlier, i.e., the space of tangent vectors $\gamma'(0)$ with $\gamma : (-\epsilon, \epsilon) \to X \subseteq \mathbf{R}^n$ and $\gamma(0) = x$. In the case of $O_n \subseteq M_n \mathbf{R}$, $x = I$, we have $\gamma(t)^t \gamma(t) = I$ for all $t \in (-\epsilon, \epsilon)$. By differentiation, we obtain $\gamma'(0)^t \gamma(0) + \gamma(0)^t \gamma'(0) = 0$, hence $\gamma'(0)^t = -\gamma'(0)$. Thus, $T_e O_n \subseteq \text{skew}_n \mathbf{R}$. Conversely, if $A \in \text{skew}_n \mathbf{R}$, let $\gamma(t) = \exp tA$. Then we have $\gamma : \mathbf{R} \to M_n \mathbf{R}$, such that $\gamma(\mathbf{R}) \subseteq O_n$ and $\gamma(0) = I$. By differentiation, $\gamma'(0) = A$. Hence $\text{skew}_n \mathbf{R} \subseteq T_e O_n$.)

The (matrix) exponential function

$$\exp A = \sum_{n \geq 0} \frac{A^n}{n!},$$

which appeared in Example (3), is very useful. (It is easy to show that the series converges for any A.) It has the following properties:

Proposition.

(1) The exponential map $\exp : M_n\mathbf{R} \to GL_n\mathbf{R}$ *is a local chart at* $I \in GL_n\mathbf{R}$.

(2) Let G *be a Lie subgroup of* $GL_n\mathbf{R}$. *Then the exponential map restricts to a map* $\mathbf{g} \to G$, *and this map is a local chart at* $I \in G$. ■

(This proposition may be proved by calculating the derivative of the exponential map at $0 \in M_n\mathbf{R}$. If f is any (smooth) function, then the derivative Df is given by the formula $(Df)_x(V) = \frac{d}{dt} f(\gamma(t))|_0$, where $V = \frac{d}{dt}\gamma(t)|_0$. Hence, $(D\exp)_0 A = \frac{d}{dt}\exp tA|_0 = A$. Thus, $(D\exp)_0$ is the identity map!)

More generally, it is possible to define $\exp : \mathbf{g} \to G$ for *any* Lie group. We shall not need the general definition. However, the following useful property of the exponential map (which is valid also in the general case) should be noted: For any $X \in \mathbf{g}$, the map $\gamma : t \mapsto \exp tX$ provides a curve in G with $\gamma(0) = e$ and $\gamma'(0) = X$.

Exercises:

(1.6) Let $\gamma, \delta : (-\epsilon, \epsilon) \to M_n\mathbf{R}$. Show that $(\gamma + \delta)' = \gamma' + \delta'$ and $(\gamma\delta)' = \gamma'\delta + \gamma\delta'$.

(1.7) Let $A, B \in M_n\mathbf{R}$. Is it true that $\exp A \exp B = \exp(A + B)$?

(1.8) Show that $T_e U_n = \text{skewHerm}_n\,\mathbf{C} = \{A \in M_n\mathbf{C} \mid A^* = -A\}$.

(1.9) Show that $T_e SL_n\mathbf{R} = \{A \in M_n\mathbf{R} \mid \text{trace } A = 0\}$.

(1.10) Show that the exponential map $\exp : \text{skewHerm}_n \to U_n$ is surjective.

Bibliographical comments.

See the comments at the end of Chapter 3.

Chapter 2: Lie Algebras

I. Definitions and examples.

Definition. *Let V be a vector space (real or complex). We say that V is a Lie algebra (real or complex) if*

(1) there is a bilinear map $[\ ,\ ] : V \times V \to V$, such that

(2) $[X,Y] = -[Y,X]$ and $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$ for all $X,Y,Z \in V$.

Examples:

(1) V = any vector space, $[X,Y] = 0$ for all $X,Y \in V$. (In this case, we say that V is *abelian*.)

(2) V = the vector space consisting of all vector fields on a manifold M, $[\ ,\ ]$ = bracket of vector fields. (Recall that the bracket of vector fields V_1, V_2 on any manifold M is defined by $[V_1,V_2]f = V_1(V_2 f) - V_2(V_1 f)$, where $f : M \to \mathbf{R}$ is any function.)

(3) $V = \mathbf{R}^3$, $[X,Y] = X \times Y$ (vector cross product).

(4) $V = \mathrm{End}(W)$, the vector space of all endomorphisms of a vector space W, $[X,Y] = XY - YX$. (If $V = \mathbf{R}^n$ or $\mathbf{C}^n$, $\mathrm{End}(V) = M_n\mathbf{R}$ or $M_n\mathbf{C}$.)

Let G be a Lie group. Let $\mathbf{g} = T_e G$. We can construct a Lie algebra structure for $\mathbf{g}$, as follows. For any $X \in \mathbf{g}$, we define a vector field X^* on G by

$$X^*_g = DL_g(X) = \tfrac{d}{dt} g \exp tX|_0.$$

(Here $L_g : G \to G$ is given by $L_g(h) = gh$. Later, we shall need the analogous map $R_g : G \to G$, $R_g(h) = hg$.) In this way, we can identify $\mathbf{g}$ with a subspace of the Lie algebra of all vector fields on G (see Example (2)). This subspace consists precisely of the *left-invariant* vector fields on G, i.e., the vector fields V such that $V_{gh} = DL_g(V_h)$ for all $g,h \in G$.

Lemma. *If U,V are left-invariant vector fields on G, then so is $[U,V]$.*

Proof. For any $f : G \to \mathbf{R}$ we have

$$V_{gh} f = V f(gh) = V f \circ L_g(h)$$
$$DL_g(V_h) f = DL_g(V) f(h) = Df(DL_g(V))(h) = V(f \circ L_g)(h),$$

so the condition $V_{gh} = DL_g(V_h)$ (for all $h \in G$) is equivalent to the condition $V f \circ L_g(h) = V(f \circ L_g)(h)$ (for all $h \in G$, $f : G \to \mathbf{R}$). If

U, V satisfy this condition, we have

$$\begin{aligned}[U,V](f \circ L_g) &= U(V(f \circ L_g)) - V(U(f \circ L_g)) \\ &= U((Vf) \circ L_g) - V((Uf) \circ L_g) \\ &= (U(Vf)) \circ L_g - (V(Uf)) \circ L_g \\ &= ([U,V]f) \circ L_g.\end{aligned}$$

So $[U, V]$ satisfies the same condition. ■

It follows that $\mathbf{g}$ inherits the structure of a Lie algebra; it becomes a subalgebra of the Lie algebra of all vector fields. We therefore obtain $[\ ,\ ] : \mathbf{g} \times \mathbf{g} \to \mathbf{g}$ (satisfying conditions (1) and (2) above). By definition we have $[X^*, Y^*] = [X, Y]^*$, for any $X, Y \in \mathbf{g}$. In future, we call $\mathbf{g}$ the *Lie algebra of the Lie group G.*

Proposition. *If G is a matrix group, then the Lie algebra structure of $\mathbf{g}$ is given by*

$$[X, Y] = XY - YX$$

(where XY denotes the product of the matrices X, Y).

Sketch of the proof. We have $X^* f(g) = \frac{d}{dt} f(g \exp tX)|_0$. Hence,

$$(Y^*(X^* f))(e) = \tfrac{d}{ds}\tfrac{d}{dt} f(\exp sY \exp tX)|_0|_0.$$

If f is linear, the proposition follows from this. The general case can be deduced from the case where f is linear. ■

Exercise:

(2.1) If $G = SO_3$ (and if $\mathbf{g}$ is identified with $\mathbf{R}^3$), show that we obtain Example (3) above.

The main significance of the Lie algebra (of a Lie group) is demonstrated by the next theorem:

Theorem. *There is a one to one correspondence between*

(1) connected Lie subgroups G of $GL_n\mathbf{R}$ (or $GL_n\mathbf{C}$), and

(2) Lie subalgebras V of $M_n\mathbf{R}$ (or $M_n\mathbf{C}$). ■

The correspondence assigns to a Lie group G its Lie algebra $\mathbf{g}$. (It follows from the definition of a Lie subgroup in Chapter 1 that this procedure is valid.) A proof of the theorem can be found in Varadarajan [1984], Theorem 2.5.2. There is a more general result, the "Fundamental Theorem of Lie Theory", which establishes a one to one correspondence between *arbitrary*

Lie groups (up to local isomorphism) and Lie algebras. Details of this may also be found in Varadarajan [1984], section 2.8.

Examples:

(1) $G = SO_n$, $V = \text{skew}_n \mathbf{R}$.

(2) For fixed $a, b \in \mathbf{R}$, with $(a,b) \neq (0,0)$, let

$$G = \left\{ \begin{pmatrix} \exp \sqrt{-1}\, at & 0 \\ 0 & \exp \sqrt{-1}\, bt \end{pmatrix} \middle| \, t \in \mathbf{R} \right\}$$

$$V = \left\{ \begin{pmatrix} \sqrt{-1}\, at & 0 \\ 0 & \sqrt{-1}\, bt \end{pmatrix} \middle| \, t \in \mathbf{R} \right\}.$$

Observe that G is isomorphic to $S^1 = U_1$ if $b = 0$ or a/b is rational, and G is isomorphic to $\mathbf{R} = M_1\mathbf{R}$ otherwise. (For comments on the topology of G, see the end of section II of Chapter I.)

(3) Let G be a Lie subgroup of $GL_n\mathbf{R}$. Hence, $\mathbf{g}$ is a (real) Lie subalgebra of $M_n\mathbf{R}$, and $\mathbf{g} \otimes \mathbf{C}$ is a (complex) Lie subalgebra of $M_n\mathbf{C}$. By the theorem, there exists a (complex) Lie subgroup G^c of $GL_n\mathbf{C}$ whose Lie algebra is $\mathbf{g} \otimes \mathbf{C}$. The complex group G^c is called the *complexification* of G. For example, $GL_n\mathbf{C}$ is the complexification of $GL_n\mathbf{R}$. *Warning: It is possible to have $G_1^c = G_2^c$ and $G_1 \neq G_2$!* For example, the complexification of U_n is also $GL_n\mathbf{C}$.

The correspondence between Lie groups and Lie algebras is extremely important because (very roughly speaking) it reduces the study of Lie groups to linear algebra. The Lie algebra is also a useful tool in the study of the differential geometry of Lie groups. As an example, we mention the following simple result.

A *Riemannian metric* $\langle\ ,\ \rangle$ on a manifold X is by definition an inner product $\langle\ ,\ \rangle_x$ on each tangent space T_xX (and it is assumed that $\langle\ ,\ \rangle_x$ depends smoothly on x). If $X = G$, a Lie group, then a *left-invariant* Riemannian metric is a Riemannian metric $\langle\ ,\ \rangle$ such that $\langle X, Y\rangle_k = \langle DL_hX, DL_hY\rangle_{hk}$ for all $X, Y \in T_kG$, and all $h, k \in G$.

Proposition. *There is a one to one correspondence between*

(1) left-invariant Riemannian metrics $\langle\ ,\ \rangle$ on G, and

(2) inner products $\langle\langle\ ,\ \rangle\rangle$ on $\mathbf{g}$. ■

(Given $\langle\ ,\ \rangle$, we define $\langle\langle\ ,\ \rangle\rangle = \langle\ ,\ \rangle_e$. Conversely, given $\langle\langle\ ,\ \rangle\rangle$, we define $\langle X, Y\rangle_g = \langle\langle DL_{g^{-1}}X, DL_{g^{-1}}Y\rangle\rangle$, where $X, Y \in T_gG$.)

II. The adjoint representation.

Definition. *Let G be a Lie group. Let V be a vector space. A representation of G on V is a homomorphism $G \to GL(V)$.*

(The notation $GL(V)$ means the group of invertible linear transformations $T : V \to V$. For example, $GL(\mathbf{R}^n) = GL_n\mathbf{R}$.)

Definition. *Let G be a Lie group, and let $\mathbf{g}$ be the Lie algebra of G. The adjoint representation of G on $\mathbf{g}$ is the homomorphism*

$$\mathrm{Ad} : G \to GL(\mathbf{g}), \quad g \mapsto D(L_g \circ R_{g^{-1}})_e.$$

Alternatively:

$$\mathrm{Ad}(g)X = \tfrac{d}{dt} g \exp tX\, g^{-1}|_{t=0}.$$

Proposition. *If G is a matrix group, then* $\mathrm{Ad}(A)X = AXA^{-1}$.

Proof. $\mathrm{Ad}(A)X = \frac{d}{dt}\ A(\exp tX)A^{-1}|_0 = \frac{d}{dt}\ \exp tAXA^{-1}|_0 = AXA^{-1}$. ■

There is a version of the adjoint representation for Lie algebras. First, a representation of a Lie algebra $\mathbf{g}$ on a vector space V is defined to be a Lie algebra homomorphism $\mathbf{g} \to \mathrm{End}(V)$. Next, the adjoint representation of $\mathbf{g}$ on $\mathbf{g}$ is defined to be the homomorphism

$$\mathrm{ad} = D(\mathrm{Ad})_e : \mathbf{g} \to \mathrm{End}(\mathbf{g}).$$

Proposition. *If G is a matrix group, then* $\mathrm{ad}(X)Y = XY - YX$.

Proof. $D(\mathrm{Ad})_e(X)Y = \frac{d}{dt}\mathrm{Ad}(\exp tX)Y|_0 = \frac{d}{dt}(\exp tX)Y(\exp -tX)|_0 = XY - YX$. ■

(More generally, for arbitrary Lie groups, it can be shown that $\mathrm{ad}(X)Y = [X,Y]$.)

From now on, we shall only consider matrix groups! This is not a serious restriction, and it simplifies our calculations.

The adjoint representation is useful in studying the geometry of G. For example:

Proposition. *There is a one to one correspondence between*

(1) bi-invariant Riemannian metrics $\langle\ ,\ \rangle$ on G (bi-invariant means left invariant and right invariant), and

(2) Ad*-invariant inner products* $\langle\langle\ ,\ \rangle\rangle$ *on* $\mathbf{g}$ *(*Ad*-invariant means that* $\langle\langle \mathrm{Ad}(g)X, \mathrm{Ad}(g)Y\rangle\rangle = \langle\langle X, Y\rangle\rangle$ *for all* $X, Y \in \mathbf{g}$ *and all* $g \in G$*).* ■

This is proved in the same way as the proposition at the end of the last section.

By "averaging over G" any inner product on $\mathbf{g}$, and using the above correspondence, it is possible to prove the following existence result:

Proposition. *If G is a compact Lie group, then there exists a bi-invariant Riemannian metric on G.* ■

For example, if $G = O_n$, an Ad-invariant inner product on $\mathrm{skew}_n\,\mathbf{R}$ is given by $\langle\langle A, B\rangle\rangle = -\mathrm{trace}\,AB$. By the proposition, we obtain a bi-invariant Riemannian metric on O_n.

A deeper property of the adjoint representation is that it measures the non-commutativity of G. For example, if G is abelian, then $\mathrm{Ad}(g)X = X$ for all $g \in G, X \in \mathbf{g}$. The next definition helps to clarify this idea.

Definition. *Let G be a compact Lie group. A Cartan subalgebra of* $\mathbf{g}$ *is a maximal abelian Lie subalgebra of* $\mathbf{g}$.

For example, let

$$\mathbf{c}_n = \left\{ \begin{pmatrix} \sqrt{-1}\,x_1 & & & \\ & \sqrt{-1}\,x_2 & & \\ & & \cdots & \\ & & & \sqrt{-1}\,x_n \end{pmatrix} \;\middle|\; x_1, \dots, x_n \in \mathbf{R} \right\}.$$

Then $\mathbf{c}_n$ is a Cartan subalgebra of $\mathrm{skewHerm}_n\,\mathbf{C}$.

Theorem (Cartan). *Let G be a compact Lie group. Let* $\mathbf{g}$ *be the Lie algebra of G. Let* $\mathbf{c}_1, \mathbf{c}_2$ *be two Cartan subalgebras of* $\mathbf{g}$*. Then there exists some $g \in G$ such that* $\mathrm{Ad}(g)\mathbf{c}_1 = \mathbf{c}_2$*.* ■

Corollary. *Let G be a compact Lie group. Let* $\mathbf{g}$ *be the Lie algebra of G. Let* $\mathbf{c}$ *be a Cartan subalgebra of* $\mathbf{g}$*. If* $X \in \mathbf{g}$*, then there exists some $g \in G$ such that* $\mathrm{Ad}(g)X \in \mathbf{c}$*.* ■

This is a "diagonalization theorem". For example, let us take $G = U_n, \mathbf{g} = \mathrm{skewHerm}_n\,\mathbf{C}$, and $\mathbf{c} = \mathbf{c}_n$. Then we obtain the following familiar fact from linear algebra: If $X \in \mathrm{skewHerm}_n\,\mathbf{C}$, then there exists some $A \in U_n$ such that $AXA^{-1} \in \mathbf{c}_n$.

There is a version of Cartan's theorem for (compact, connected) Lie groups, where the concept of "Cartan subalgebra" is replaced by the concept of "maximal torus". As a corollary, we obtain diagonalization theorems for Lie groups. The appropriate definition is: