1
Formulation of physical problems

For an engineer or a physical scientist, the first necessary skill in doing theoretical analysis is to describe a problem in mathematical terms. To begin with, one must make use of the basic laws that govern the elements of the problem. In continuum mechanics, these are the conservation laws for mass, momentum (Newton's second law), energy, etc. In addition, empirical constitutive laws are often needed to relate certain unknown variables; examples are Hooke's law between stress and strain, Fourier's law between heat flux and temperature, and Darcy's law between seepage velocity and pore pressure.

To derive the conservation law one may consider an infinitesimal element (a line segment, area, or volume element), yielding a differential equation directly. Alternatively, one may consider a control volume (or area, or line segment) of arbitrary size in the medium of interest. The law is first obtained in integral form; a differential equation is then derived by using the arbitrariness of the control volume. The two approaches are completely equivalent.

Let us first demonstrate the differential approach.

1.1 Transverse vibration of a taut string

Referring to Figure 1.1, we consider a taut string stretched between two fixed points at $x = 0$ and $x = L$ and displaced laterally by a distribution of external force. Conservation of transverse momentum requires that the total lateral force on the string element be balanced by its inertia. Let the lateral displacement be $V(x, t)$ and consider a differential element between x and $x + dx$. The net transverse force due to the difference of

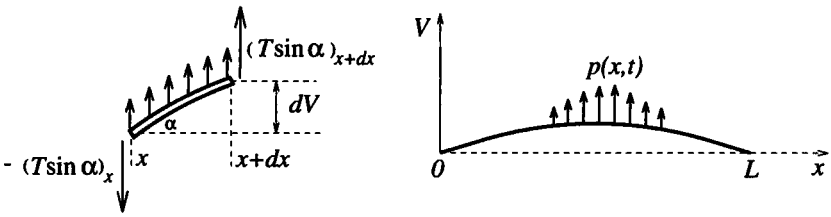

Fig. 1.1. Deformation of a taut string.

tension at both ends of the element is

$$(T\sin\alpha)_{x+dx} - (T\sin\alpha)_x\,,$$

where T denotes the local tension in the string and

$$\sin\alpha = \frac{dV}{\sqrt{dx^2+dV^2}} = \frac{\frac{\partial V}{\partial x}}{\sqrt{1+(\frac{\partial V}{\partial x})^2}}.$$

We shall assume the lateral displacement to be small everywhere so that the slope is also small: $\frac{\partial V}{\partial x} \ll 1$. The local value of $\sin\alpha$ can then be approximated by

$$\frac{\partial V}{\partial x} + O\left(\frac{\partial V}{\partial x}\right)^3,$$

where the expression $O(\delta)$ stands for *of the order of* δ. For any smooth function f, Taylor expansion gives

$$f(x+dx) - f(x) = \left(\frac{\partial f}{\partial x}\right)dx + O(dx)^2\,,$$

where the derivative is evaluated at x. Hence the net vertical force is

$$\frac{\partial}{\partial x}\left(T\frac{\partial V}{\partial x}\right)dx + O(dx)^2.$$

For infinitesimal stretching the string tension is proportional to the strain, according to Hooke's law. The initial strain is $\Delta L/L$ so that the initial tension is

$$T = ES\frac{\Delta L}{L},$$

where E denotes Young's modulus of elasticity and S denotes the cross-sectional area of the string. For simplicity, both E and S, hence T, will be assumed to be uniform in x. With lateral displacement, the strain

must be changed. Consider the part of the string extending from 0 to x. The length $\ell(x,t)$ of this deformed part is

$$\ell(x,t) = \int_0^x dx \left[1 + \left(\frac{\partial V}{\partial x}\right)^2\right]^{1/2} = x\left[1 + O\left(\frac{\partial V}{\partial x}\right)^2\right],$$

hence the corresponding strain is

$$\frac{\ell - x}{x} = O\left(\frac{\partial V}{\partial x}\right)^2 \quad \text{for all} \quad 0 < x < L,$$

which is of second-order smallness. As long as $(\ell - x)/x = O[(\ell - L)/L] \ll \Delta L/L$, the tension is essentially unchanged, i.e., T is indistinguishable from its initial constant value. Thus the net vertical force on the string element is well represented by

$$T\frac{\partial^2 V}{\partial x^2}dx.$$

If the mass per unit length of the string is ρ, the inertia of the element is $\rho(\partial^2 V/\partial t^2)dx$. Let the applied load per unit length be $p(x,t)$. Balancing forces and inertia we get

$$\rho dx\frac{\partial^2 V}{\partial t^2} = T\frac{\partial^2 V}{\partial x^2}dx + pdx + O\left(dx\right)^2.$$

Eliminating dx and taking the limit of $dx \to 0$, we get

$$\frac{\rho}{T}\frac{\partial^2 V}{\partial t^2} - \frac{\partial^2 V}{\partial x^2} = \frac{p}{T}. \tag{1.1.1}$$

This equation, called the *wave equation*, is a partial differential equation of the second order. It is linear in the unknown V and inhomogeneous because of the forcing term on the right-hand side.

In (1.1.1) the highest derivatives with respect to x and t are of second order, hence we need two boundary conditions, one at $x = 0$ and one at $x = L$, and two initial conditions at $t = 0$. For example, if the ends of the string are fixed, then

$$V(0,t) = V(L,0) = 0. \tag{1.1.2}$$

If the displacement and velocity are known at $t = 0$, then

$$V(x,0) = f(x), \qquad \frac{\partial V}{\partial t}(x,0) = g(x). \tag{1.1.3}$$

Equations (1.1.1–1.1.3) constitute the initial-boundary-value problem for $V(x,t)$ and complete the formulation.

Is the longitudinal displacement U along the x direction important in this problem? Conservation of momentum in the x direction requires that

$$\rho dx \frac{\partial^2 U}{\partial t^2} = (T\cos\alpha)_{x+dx} - (T\cos\alpha)_x .$$

Since

$$\cos\alpha = \frac{dx}{\sqrt{(dx)^2 + (dV)^2}} = \frac{1}{\sqrt{1 + \left(\frac{\partial V}{\partial x}\right)^2}} \cong 1 + O\left(\frac{\partial V}{\partial x}\right)^2 ,$$

the acceleration is of second-order smallness

$$\frac{\rho}{T}\frac{\partial^2 U}{\partial t^2} = O\left(\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right)^2\right) = O\left(\left(\frac{\partial V}{\partial x}\right)\frac{\rho}{T}\frac{\partial^2 V}{\partial t^2}\right).$$

Thus the longitudinal motion is negligible in comparison.

1.2 Longitudinal vibration of an elastic rod

Consider an elastic rod with the cross-sectional area $S(x)$ and Young's modulus E, as shown in Figure 1.2.

Let the longitudinal displacement from equilibrium be $U(x,t)$. The strain at station x is

$$\lim_{\Delta x \to 0} \frac{\Delta U}{\Delta x} = \frac{\partial U}{\partial x}.$$

By Hooke's law, the tension at x is

$$ES\frac{\partial U}{\partial x}.$$

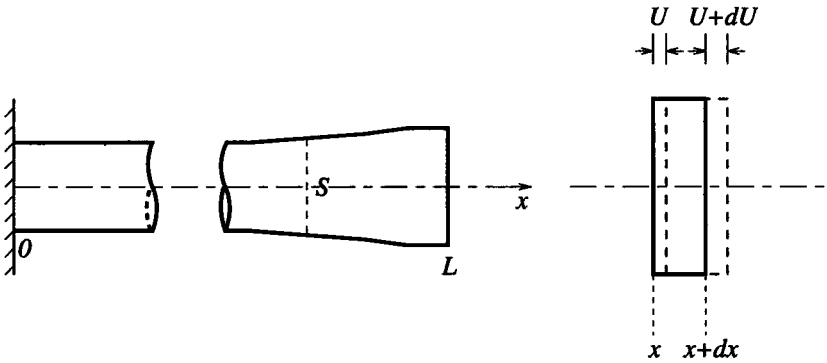

Fig. 1.2. Longitudinal deformation of an elastic rod.

Now the net tension on a rod element from x to $x + dx$ is

$$\left(ES\frac{\partial U}{\partial x}\right)_{x+dx} - \left(ES\frac{\partial U}{\partial x}\right)_{x} = dx\frac{\partial}{\partial x}\left(ES\frac{\partial U}{\partial x}\right) + O\left(dx\right)^2$$

Let the externally applied longitudinal force be $F(x,t)$ per unit length. Momentum conservation requires that

$$\rho S\frac{\partial^2 U}{\partial t^2}dx = \frac{\partial}{\partial x}\left(ES\frac{\partial U}{\partial x}\right)dx + Fdx + O(dx)^2.$$

In the limit of vanishing dx, we get the differential equation

$$\rho S\frac{\partial^2 U}{\partial t^2} = \frac{\partial}{\partial x}\left(ES\frac{\partial U}{\partial x}\right) + F. \tag{1.2.1}$$

In the special case of uniform cross section, S = constant, and U satisfies the inhomogeneous wave equation

$$\frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 U}{\partial x^2} + \frac{F}{ES}, \tag{1.2.2}$$

where $c = \sqrt{E/\rho}$ has the dimension of velocity.

The simplest boundary conditions are for fixed or free ends. If both ends are fixed, then

$$U(0,t) = 0 \qquad \text{and} \qquad U(L,t) = 0. \tag{1.2.3}$$

If the left end is fixed but the right end is free, then we have instead

$$U(0,t) = 0 \qquad \text{and} \qquad \frac{\partial U}{\partial x}(L,0) = 0, \tag{1.2.4}$$

since the stress is proportional to the strain. Again, the most natural initial conditions are

$$U(x,0) = f(x) \qquad \text{and} \qquad \frac{\partial U}{\partial t}(x,0) = g(x), \tag{1.2.5}$$

where f and g are prescribed functions of x for $0 < x < L$.

There are practical situations where the boundary conditions are not so simple. For example, consider a heavy mass M of small size attached to the end $x = L$, which is otherwise free, as shown in Figure 1.3. Momentum conservation of the mass M requires that

$$M\frac{\partial^2 U}{\partial t^2} + ES\frac{\partial U}{\partial x} = 0, \quad x = L. \tag{1.2.6}$$

This equation serves as a boundary condition for the rod, now involving both $\partial^2 U/\partial t^2$ and $\partial U/\partial x$ at the end.

Let us change to the integral approach in the next example.

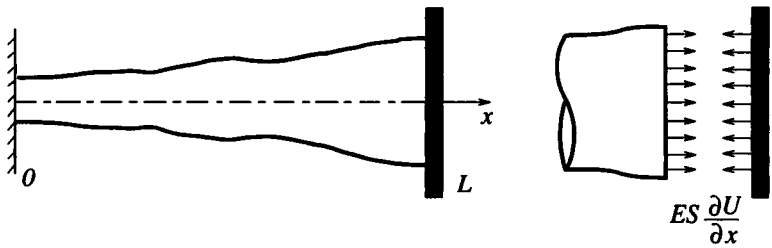

Fig. 1.3. A heavy mass at the end of a rod.

1.3 Traffic flow on a freeway

One of the mathematical models of traffic flow is the hydrodynamical theory of Lighthill and Whitham (1955). It is a simple theory capable of describing many real-life features of highway traffic with remarkable accuracy. Consider any section of a straight freeway from $x = a$ to $x = b$; see Figure 1.4. Assume for simplicity that there are no exits or entrances, and all vehicles are on the move. Let the density of cars (number of cars per unit length of highway) at x and t be $\rho(x,t)$, and the flux of cars (number of cars crossing the point x per unit time) be $q(x,t)$. By requiring that the number of cars within an arbitrary section from a to b be conserved, we have

$$-\frac{\partial}{\partial t}\int_a^b \rho(x,t)\,dx = q(b,t) - q(a,t).$$

Rewriting the right-hand side

$$q(b,t) - q(a,t) = \int_a^b \frac{\partial q}{\partial x}\,dx,$$

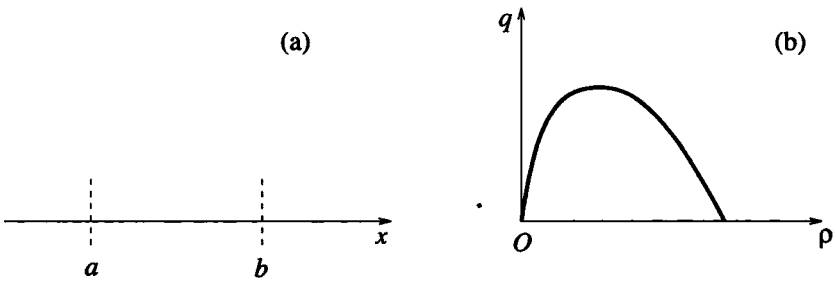

Fig. 1.4. (a) A section of the freeway. (b) The relation between traffic flux rate and traffic density.

we get

$$\int_a^b \left(\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} \right) dx = 0. \tag{1.3.1}$$

Since the control interval (a, b) is arbitrary, the integrand must vanish,

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0. \tag{1.3.2}$$

This result can be argued by contradiction, which is a typical reasoning needed to change an integral law to a differential law. Suppose the integrand is positive somewhere within (a, b), say in the range $(a', b') \in (a, b)$, and zero elsewhere in (a, b), then the integral in (1.3.1) must be positive. But this is a contradiction. The assumption that the integrand is positive somewhere is therefore wrong. By a similar argument, the integrand cannot be negative anywhere and hence must be zero everywhere in (a, b).

Equation (1.3.2) is the canonical differential form of all conservation laws. Having two unknowns q and ρ, a constitutive relation between ρ and q is needed and must be found by field measurements. Heuristically, q must be zero when there is no car on the road and zero again when the density attains a maximum (bumper-to-bumper traffic), hence the relation between q and ρ must be nonlinear

$$q = q(\rho), \tag{1.3.3}$$

as sketched in Figure 1.4b. With this relation, (1.3.2) becomes

$$\frac{\partial \rho}{\partial t} + \left(\frac{dq}{d\rho} \right) \frac{\partial \rho}{\partial x} = 0. \tag{1.3.4}$$

This result is a first-order nonlinear partial differential equation. As a sample initial-boundary-value problem, we may specify $\rho(x, 0) = f(x)$, where $f(x)$ vanishes outside a finite range of x, and $\rho(x, t) \to 0$ as $x \to \pm\infty$.

1.4 Seepage flow through a porous medium

Soil is a porous medium consisting of densely packed grains with fluid filling the interstitial pores. Bypassing the complicated details on the granular scale, one is usually interested only in averages over volumes or areas much larger than the size of the grains but much smaller than the scales typical of the velocity or pressure variations. Let us examine a steady seepage flow and define the seepage velocity $\mathbf{q}(x, y, z)$ to be the

averaged local rate of fluid volume flowing through a unit area of the soil interior. If there is no source or sink the fluid mass in an arbitrary volume $\mathcal{V}$ must be conserved. For an incompressible fluid the net outflux through the bounding surface $\mathcal{S}$ must therefore vanish,

$$\iint_{\mathcal{S}} \mathbf{q} \cdot \mathbf{n}\, d\mathcal{S} = 0.$$

By Gauss' theorem, which is reviewed in Appendix A, the surface integral can be changed to a volume integral so that

$$\iiint_{\mathcal{V}} \nabla \cdot \mathbf{q}\, d\mathcal{V} = 0. \tag{1.4.1}$$

Now since the volume $\mathcal{V}$ is arbitrary, the integrand must be zero everywhere in the medium, thus

$$\nabla \cdot \mathbf{q} = 0. \tag{1.4.2}$$

For an isotropic and homogeneous soil there is an empirical law of momentum conservation relating $\mathbf{q}$ to the pore pressure p,

$$\mathbf{q} = -k\nabla\left(\frac{p}{\rho g} + y\right), \tag{1.4.3}$$

where ρ is the density of the pore fluid and g the gravitational acceleration. The empirical coefficient k is called the hydraulic conductivity, which is a constant if the porous medium is uniform and isotropic. Equation (1.4.3) is called *Darcy's law*, whose theoretical basis can be derived by selecting models on the granular scale, as will be described in a later chapter. It is convenient to define the velocity potential ϕ by

$$\phi = -k\left(\frac{p}{\rho g} + y\right) \tag{1.4.4}$$

such that Darcy's law becomes

$$\mathbf{q} = \nabla\phi. \tag{1.4.5}$$

Using this law in (1.4.2), we get

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0, \tag{1.4.6}$$

which is called the *Laplace equation.*

There can be a large variety of boundary conditions on the macro scale. For example, if the soil/water interface is immersed in water at

the depth $y = -h(x, z)$ measured from the water surface, the pressure along the interface is

$$p = p_o + \rho g h(x, z), \qquad y = -h(x, z),$$

where p_o is the atmospheric pressure. In terms of the velocity potential the boundary condition is

$$\phi = -\frac{kp_o}{\rho g}, \qquad y = -h(x, z). \tag{1.4.7}$$

If the soil rests on an impervious rock, the normal velocity must vanish so that

$$\frac{\partial \phi}{\partial n} = 0. \tag{1.4.8}$$

Equation (1.4.7), which prescribes the unknown function itself, is called a *Dirichlet condition*, and (1.4.8), which prescribes the normal derivative of the unknown, is called a *Neumann condition*. In some seepage problems, however, Dirichlet condition is specified along a part of the soil boundary, while Neumann condition is specified on the remaining part. Then the boundary condition is of the mixed type, and the problem is not easy.

Still more challenging is the class of free-surface problems in which there is a water table (phreatic surface) whose location is unknown *a priori*. Two boundary conditions are needed. One is the kinematic condition that the flow must be tangential to the water table. The other is the dynamical condition that the pore pressure be prescribed (zero, if capillarity is ignored). Figure 1.5 depicts the two-dimensional cross section of a long earth dam resting on a horizontal and impermeable river bed. On the left side is a reservoir of water depth H; on the right side the river is dry. On the bottom $BCDE$ the flow must be tangential to the river bed so that

$$\frac{\partial \phi}{\partial n} = 0. \tag{1.4.9}$$

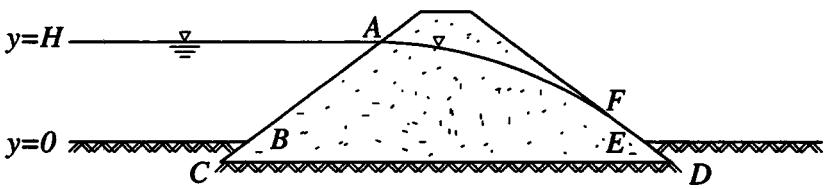

Fig. 1.5. Seepage flow through an earth dam.

On the submerged sloping surface AB, the pressure is known

$$p = p_o + \rho g(H - y), \qquad y = F(x),$$

or

$$\phi = -k\left(\frac{p}{\rho g} + y\right) = -k\left(\frac{p_o}{\rho g} + H\right) = \text{constant}, \quad y = F(x). \tag{1.4.10}$$

On the phreatic surface $y = \eta(x)$, the kinematic condition is

$$\frac{\partial \phi}{\partial n} = 0, \qquad y = \eta(x), \tag{1.4.11}$$

while the dynamic condition is $p = p_o$, or

$$\phi + k\eta = -k\left(\frac{p_o}{\rho g}\right) = \text{constant}, \qquad y = \eta(x). \tag{1.4.12}$$

Finally, on the exposed sloping surface EF at the known position $y = G(x)$, $p = p_o$ so that

$$\phi = -k\left[\frac{p_o}{\rho g} + G(x)\right]. \tag{1.4.13}$$

The set of boundary conditions (1.4.9–1.4.13) is not only mixed but also nonlinear because the position of the water table is a part of the unknown solution. This type of problem is called a *free-boundary* problem.

1.5 Diffusion in a stationary medium

Diffusion is important in many different physical contexts. Dye can diffuse in water, dust in air, heat in solids, and pollutants in the atmosphere, lakes, rivers and oceans, etc. Whatever the cause, the process is often governed by one special type of partial differential equation.

Consider the temperature $T(x, y, z, t)$ in a homogeneous solid. By energy conservation, the rate of increase of internal energy must be equal to the sum of the rate of energy influx through the bounding surface and the rate of internal energy production.

The energy outflux through a closed geometrical surface S in the solid is

$$\iint_S \mathbf{q} \cdot \mathbf{n}\, dS.$$

Using Gauss' theorem, we can rewrite the total outflux of energy as

$$\iint_S \mathbf{q} \cdot \mathbf{n}\, dS = \iiint_V \nabla \cdot \mathbf{q}\, dV.$$