PREFACE

The James space J and the James tree space JT were constructed by Robert C. James in 1950 and 1974 respectively, to answer negatively several long standing conjectures in Banach space theory regarding the reflexivity of Banach spaces with enough good properties, such as for example having a basis or a separable dual. Since then these spaces have proved to be counterexamples to many other conjectures and have been the cornerstone for constructing other spaces which have enriched the wealth of existing Banach spaces.

On the other hand, the study of their inherent properties has created new branches in the geometry of Banach spaces, leading to the development of diverse topics such as the theory of quasi-reflexive spaces and the Banach spaces based on binary and other trees; the list of references in the bibliography, exceeding 100 titles, gives an indication of the vast amount of work devoted to the study of the subject, which nonetheless is far from exhausted.

Yet, to the best of our knowledge, a unified account of the theory of James spaces is still lacking. Therefore we think that a monograph on these spaces may prove to be useful for the students of these matters.

Given the size of the subject, a completely self-contained and exhaustive exposition seems impossible; hence a selection of the material was unavoidable. We chose to concentrate on the most classical papers dealing with James spaces; however, for the sake of completeness, we give a brief account of most of the new results in the last two chapters. Also, we intend this work to be accessible to graduate students, and it is for this reason that the proofs we do not give here are to be found in well known books. On the other hand, most of the proofs that are given here come from the original papers, because we feel that this may help the reader to go back to the original sources, although in some instances we make use of later works to simplify them. In every case it is indicated where the proofs can be found, and to spare the reader unnecessary work, a serious effort was made to give enough details so that the arguments can be followed easily.

The book is organized as follows:

In chapter one we specify the prerequisites for reading the book and mention some theorems needed later on. Most of the material of this chapter is now classical and can be found for instance in treatises by Beauzamy [1], Day [1], Lindenstrauss and Tzafriri [1] and Singer [1], cited in the references.

Chapters two and three are the core of the book.

Chapter two is devoted to the study of the James space J and its dual J^*. Here we discuss their most basic characteristics, such as the quasi-reflexivity or their complemented subspaces, giving complete proofs of almost every statement. Among the subjects covered in this chapter we mention the Banach-Saks property, the spreading models of J and the type and cotype of both spaces. We also introduce the important space $\mathfrak{J} = (\sum_n J_n)_{\ell_2}$ and its properties. The chapter includes two appendices with results not directly related to the James spaces, but necessary to complement Sections 2.c and 2.i.

Chapter three is dedicated to a similar study of the James tree space JT. The topics covered in this chapter include the somewhat reflexivity, the primarity and the fixed point property of the norm in JT, as well as the Kadec-Klee property of the norm of JT^*.

The last two chapters are included mainly for completeness, but also as a reference for further study.

In chapter four we state some other results about J and JT that did not fit into the body of chapters two and three, either because of their complexity or because they required much additional elaboration. Also we give a summary of some generalizations of J and JT which have appeared in the literature through the years.

In chapter five we talk about other pathological spaces and their properties. These spaces are not directly related to those of James, but have also been created to solve some of the many questions that have arisen in the geometry of Banach spaces, and are included here so that the reader can get an overview of how things stand today.

The results in these two chapters are included mostly without proofs, because as mentioned, they would have required too much additional material or else they are out of the scope of this monograph; however, for the interested reader, full references for this material are included in the bibliography.

Although we did our best to include all of the relevant results, we are well aware that this monograph only gives a part of the story, determined by our own preferences, knowledge and understanding of the subject, but we hope that the material included in this work gives a good idea of the many and important applications of J and JT in the geometry of Banach spaces.

We would like to thank Professor Robert C. James for his encouraging support and advice, Professor Bernard Beauzamy for his valuable criticism of this monograph and our colleague Fausto Ongay for his worthwhile suggestions.

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CHAPTER 1. PRELIMINARIES

Anfang, bedenk' das Ende!

Kurfürst Georg Wilhelm von Brandenburg

The object of this chapter is to mention the prerequisites necessary for understanding this monograph. All the material in this chapter is classical and is included here for the sake of easy reference, but more complete expositions can be found in the books *Classical Banach Spaces I* by Lindenstrauss and Tzafriri [1], *Introduction to Banach Spaces and their Geometry* by Beauzamy [1] and in *Bases in Banach Spaces I* by Singer [1], to which the reader is referred for more details and proofs.

Besides a course on functional analysis, as constituted for instance by the first five chapters of Rudin [1], the reader will need some basic knowledge on classical Banach spaces and on Schauder bases; specifically he needs to know the properties of shrinking, boundedly complete and unconditional bases, as well as of block basic sequences. To fix some notations and terminology we now recall these definitions:

Definition 1.1. A sequence $\{x_n\}_{n=1}^{\infty}$ in a Banach space X is called a Schauder basis or simply a basis of X if for every $x \in X$ there is a unique sequence of scalars $\{a_n\}_{n=1}^{\infty}$ such that $x = \sum_{n=1}^{\infty} a_n x_n$.

A sequence $\{x_n\}_{n=1}^{\infty}$ which is the Schauder basis of its closed linear span is called a basic sequence.

A basis $\{x_n\}_{n=1}^{\infty}$ of a Banach space X is called unconditional if for every $x \in X$ its expansion $x = \sum_{n=1}^{\infty} a_n x_n$ in terms of the basis converges unconditionally.

The projections $\mathcal{P}_n : X \longrightarrow X$ defined by $\mathcal{P}_n(\sum_{i=1}^{\infty} a_i x_i) = \sum_{i=1}^{n} a_i x_i$ are called the natural projections associated to $\{x_n\}_{n=1}^{\infty}$ and the number $\sup_n \|\mathcal{P}_n\|$, which is finite, is called the basis constant of $\{x_n\}_{n=1}^{\infty}$.

A basis $\{x_n\}_{n=1}^{\infty}$ with basis constant one is called monotone.

Let $\{x_n\}_{n=1}^{\infty}$ be a basis of X. The functionals $x_n^* : X \longrightarrow \mathbb{R}$ given by

$$x_n^*(x_m) = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

for every $m,n \in \mathbb{N}$ are called the biorthogonal functionals associated to the basis $\{x_n\}_{n=1}^{\infty}$.

Definition 1.2. A basis $\{x_n\}_{n=1}^{\infty}$ of a Banach space X is called boundedly complete, if for every sequence of scalars $\{a_n\}_{n=1}^{\infty}$ such that

$$\sup_k \left\|\sum_{n=1}^{k} a_n x_n\right\| < \infty,$$

the series $\sum_{n=1}^{\infty} a_n x_n$ converges.

Definition 1.3. Let $\{x_n\}_{n=1}^{\infty}$ be a basis of a Banach space X. If for every $x^* \in X^*$, the norm of $x^*|_{[x_i]_{i=n}^{\infty}}$ tends to zero as n tends to infinity, the basis is called shrinking. Here X^* denotes the dual space of X and $x^*|_{[x_i]_{i=n}^{\infty}}$ the restriction of x^* to the closed linear span $[x_i]_{i=n}^{\infty}$ of $\{x_i\}_{i=n}^{\infty}$.

It can be shown that a basis $\{x_n\}_{n=1}^{\infty}$ is shrinking if and only if the biorthogonal functionals $\{x_n^*\}_{n=1}^{\infty}$ form a basis of X^*.

Definition 1.4. Let $\{x_n\}_{n=1}^{\infty}$ be a basic sequence in a Banach space X. A sequence of non-zero vectors $\{u_j\}_{j=1}^{\infty}$ in X of the form

$$u_j = \sum_{n=p_j+1}^{p_{j+1}} a_n x_n$$

with $\{a_n\}_{n=1}^{\infty}$ scalars and $0 \leq p_1 < p_2 < \ldots$ an increasing sequence of integers, is called a block basis of $\{x_n\}_{n=1}^{\infty}$.

It is easy to see that a block basis of $\{x_n\}_{n=1}^{\infty}$ is indeed a basic sequence in X and that its basis constant is less than or equal to the basis constant of $\{x_n\}_{n=1}^{\infty}$.

We will now state the basic results that will be used most often in the text. These are enunciated mostly without proof, but for a few exceptions, notably Corollary 1.9 which is one of the key tools we will

use in the sequel.
We start with a result about spaces with shrinking bases. In this case the following important theorem gives a nice and useful way to represent their double dual.

Theorem 1.5. Let $\{x_n\}_{n=1}^{\infty}$ be a shrinking monotone basis of a Banach space X and $\{x_n^*\}_{n=1}^{\infty}$ the biorthogonal functionals associated to $\{x_n\}_{n=1}^{\infty}$. Then X^{**} can be identified with the space of all sequences of scalars $\{a_n\}_{n=1}^{\infty}$ such that

$$\sup_n \|\textstyle\sum_{i=1}^{n} a_i x_i\| < \infty.$$

This correspondence for every $x^{**} \in X^{**}$ is given by

$$x^{**} \longleftrightarrow (x^{**}(x_1^*),\ x^{**}(x_2^*), \ldots).$$

The norm of x^{**} is equal to

$$\|x^{**}\| = \sup_n \|\textstyle\sum_{i=1}^{n} x^{**}(x_i^*) x_i\|.$$

If $\{x_n\}_{n=1}^{\infty}$ is a shrinking basis then it is easily seen that the set of biorthogonal functionals associated to $\{x_n\}_{n=1}^{\infty}$ is a boundedly complete basis in X^*. The converse of this is also true:

Theorem 1.6. A Banach space X with a (monotone) boundedly complete basis $\{x_n\}_{n=1}^{\infty}$ is (isometrically) isomorphic to the dual of the subspace $Z = [x_n^*]_{n=1}^{\infty}$ of X^*, where $\{x_n^*\}_{n=1}^{\infty}$ is the set of biorthogonal functionals associated to $\{x_n\}_{n=1}^{\infty}$, via the isomorphism $T : X \longrightarrow Z^*$ given by $Tx(z) = z(x)$.

The following theorem classifying reflexive spaces with an unconditional basis is due to James [1].

Theorem 1.7. Let Y be a closed subspace of a Banach space X with an unconditional basis. Then Y is reflexive if and only if Y contains no subspace isomorphic to c_0 or ℓ_1.

Since the James space J neither is reflexive nor contains any subspace isomorphic to c_0 or ℓ_1 (Theorem 2.a.2), this shows that the condition of the existence of an unconditional basis is essential for the conclusion of the previous theorem.

Proposition 1.8, due to Krein, Milman and Rutman, shows that basic sequences are stable in the following sense: if a sequence is close enough to a given basic sequence in a Banach space, the perturbed sequence is also basic and equivalent to the original one, and if the space spanned by the first sequence is complemented, so is the second.

Proposition 1.8. Let $\{x_n\}_{n=1}^{\infty}$ be a basic sequence in a Banach space X with basis constant K such that for every n = 1, 2,...

$$\frac{1}{M} \leq \|x_n\| \leq M.$$

(i) Let $\{y_n\}_{n=1}^{\infty}$ be a sequence in X with

$$\sum_{n=1}^{\infty}\|x_n - y_n\| < \frac{1}{2KM}.$$

Then $\{y_n\}_{n=1}^{\infty}$ is also a basic sequence in X which is equivalent to $\{x_n\}_{n=1}^{\infty}$ via an isomorphism T with $\|T\| < 2$.

(ii) Assume that there is a projection P from X onto the closed linear span of $\{x_n\}_{n=1}^{\infty}$ which will be denoted by $[x_n]_{n=1}^{\infty}$. Let $\{y_n\}_{n=1}^{\infty}$ be a sequence in X with

$$\sum_{n=1}^{\infty}\|x_n - y_n\| < \frac{1}{2KM\|P\|}.$$

Then $Y = [y_n]_{n=1}^{\infty}$ is complemented in X.

We will prove the next corollary, which is an important application of the previous theorem, since it will be used several times in the course of the text.

Recall that a sequence $\{x_n\}_{n=1}^{\infty}$ is seminormalized if there exists a constant M such that $\frac{1}{M} \leq \|x_n\| \leq M$ and normalized if $\|x_n\| = 1$.

Corollary 1.9. Let X be a Banach space with a normalized basis $\{x_n\}_{n=1}^{\infty}$ and let $\{y_n\}_{n=1}^{\infty}$ be a seminormalized sequence in X such that for every $m \in \mathbb{N}$ the sequence $\{x_m^*(y_n)\}_{n=1}^{\infty}$ converges to zero. Then there exists a block basic sequence $\{y'_n\}_{n=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ which is equivalent to a sub-sequence of $\{y_n\}_{n=1}^{\infty}$.

Proof: Suppose that $\frac{1}{M} \leq \|y_n\| \leq M$, that K is the basis constant of $\{x_n\}_{n=1}^{\infty}$ and that $y_n = \sum_{i=1}^{\infty} b_i^n x_i$. Let $0 < \varepsilon < \frac{1}{2KM}$ and let $0 < \varepsilon_i$ such that $\sum_{i=1}^{\infty} \varepsilon_i < \varepsilon$. We will proceed by induction:
Let $p_1 = 0$ and $n_1 = 1$. There exists p_2 such that

$$\|y_1 - \sum_{i=1}^{p_2} b_i^1 x_i\| < \varepsilon_1.$$

Let $y'_1 = \sum_{i=1}^{p_2} b_i^1 x_i$. Suppose we have constructed $y'_1, \ldots, y'_{k-1}$, $p_1 < \ldots < p_k$ and $n_1 < \ldots < n_{k-1}$ such that for $j = 1, \ldots, k-1$

$$y'_j = \sum_{i=p_j+1}^{p_{j+1}} b_i^{n_j} x_i \text{ and } \|y_{n_j} - y'_j\| < \varepsilon_j.$$

Let $n_k > n_{k-1}$ be such that $\sum_{i=1}^{p_k} |b_i^{n_k}| < \varepsilon_k/2$. This can be done because $\{x_m^*(y_n)\}_{n=1}^{\infty}$ converges to zero for $m = 1, \ldots, p_k$. Now let $p_{k+1} > p_k$ be such that

$$\|\sum_{i=p_{k+1}+1}^{\infty} b_i^{n_k} x_i\| < \varepsilon_k/2.$$

Define $y'_k = \sum_{i=p_k+1}^{p_{k+1}} b_i^{n_k} x_i$. Then $\|y_{n_k} - y'_k\| < \varepsilon_k$ and by Proposition 1.8 we are done.

Another result often used in this work is the following proposition about the associativity and other properties of the ℓ_2 direct sum; for the proof, the reader is referred to Singer [1] and Beauzamy [1].
Recall that if $\{X_n\}_{n=1}^{\infty}$ is a sequence of Banach spaces then

$$(\sum_n X_n)_{\ell_2} = \{x = \{x_n\} : x_n \in X_n \text{ and } \sum_{n=1}^{\infty} \|x_n\|_{X_n}^2 < \infty\},$$

with the norm $\|x\| = (\sum_{n=1}^{\infty} \|x_n\|_{X_n}^2)^{1/2}$.

Proposition 1.10. (a) Let X and Y be Banach spaces; then

$$(\sum_n X \oplus Y)_{\ell_2} \approx (\sum_n X)_{\ell_2} \oplus (\sum_n Y)_{\ell_2},$$

where X ≈ Y means that X is isomorphic to Y.

(b) If $\{X_n\}$ and $\{Y_n\}$ are sequences of Banach spaces such that $X_n \approx Y_n$ and there is a constant $C \geq 1$ such that for $n = 1, 2, \ldots$, there are onto isomorphisms $T_n : X_n \longrightarrow Y_n$ with $\|T_n\|\|T_n^{-1}\| \leq C$, then there exists an onto isomorphism

$$T : (\sum_n X_n)_{\ell_2} \longrightarrow (\sum_n Y_n)_{\ell_2}$$

with $\|T\|\,\|T^{-1}\| \le C$.

(c) If $\{X_n\}$ and $\{Y_n\}$ are sequences of Banach spaces such that Y_n is a closed subspace of X_n and there is a constant $C \ge 1$ such that for $n = 1, 2, \ldots$ there are onto projections $P_n : X_n \longrightarrow Y_n$ with $\|P_n\| \le C$, then there exists an onto projection $P : (\sum_n X_n)_{\ell_2} \longrightarrow (\sum_n Y_n)_{\ell_2}$ with $\|P\| \le C$.

(d) If $\{X_n\}$ is a sequence of Banach spaces then $(\sum_n X_n)^*_{\ell_2} \approx (\sum_n X^*_n)_{\ell_2}$.

In general a space X is not complemented in X^{**}; however, the dual of every Banach space X is complemented in X^{***}; to see this, we first need the following definition:

Definition 1.11. Let X be a Banach space, let Y be a subspace of X and let Z be a subspace of X^*. We define

$$Y^{\perp} = \{f \in X^* : fy = 0 \quad \text{for every} \quad y \in Y\}$$

and

$$Z_{\perp} = \{x \in X : gx = 0 \quad \text{for every} \quad g \in Z\}.$$

Lemma 1.12. Let X be a Banach space and let j_X denote the canonical embedding of X in X^{**}, then

$$X^{***} = j_{X^*}(X^*) \oplus (j_X(X))^{\perp}.$$

Proof: Let $P : X^{***} \longrightarrow j_{X^*}(X^*)$ be defined as $P = j_{X^*} \circ (j_X)^*$, where $(j_X)^*$ denotes the transpose of j_X. Then it is easy to see that $(j_X)^* \circ j_{X^*} = Id_{X^*}$ and hence P is an onto projection satisfying $(I - P)X^{***} = (j_X(X))^{\perp}$.

The fact that the James space J is of codimension one in its double dual, as will be shown in Chapter two, is one of its most outstanding features. A generalization of this property, called quasi-reflexivity, is presented in Definition 1.13, and Theorem 1.14, due to Civin and Yood

[1], states that, similarly to the property of reflexivity, a space is quasi-reflexive if and only if its dual shares this characteristic.

Definition 1.13. A Banach space X is said to be quasi-reflexive (of order k) if the quotient of X^{**} by the natural image of X in X^{**} has finite dimension (dimension k).

The existence of such spaces for every k was proved by Singer [2] and will be shown in Theorem 2.a.4.

Theorem 1.14. A Banach space X is quasi-reflexive of order n if and only if X^* is quasi-reflexive of order n.

Proof: By Lemma 1.12 $X^{***}/j_{X^*}(X^*)$ is isomorphic to $(j_X(X))^\perp$ and it is a well known fact (see e.g. Beauzamy [1]) that $(j_X(X))^\perp$ is isomorphic to $(X^{**}/j_X(X))^*$.