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E. B. DAVIES
Department of Mathematics, King's College, Strand, London

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Preface

The theory of differential equations is one of the outstanding creations of the human mind. Its influence upon the development of physical science would be hard to exaggerate. The long history and many applications of the theory, however, make it almost impossible to write a balanced account of the subject. Thus authors of student texts are confronted with the choice between writing rather superficially on a range of topics or in more depth on some narrow field, in which they have a particular interest.

In this book I have given a simple introduction to the spectral theory of linear differential operators. This spectral theory is an outgrowth of fundamental work of David Hilbert between 1900 and 1910 on the analysis of integral operators on infinite-dimensional spaces – now called Hilbert spaces. However, like almost every important new development in mathematics, it was preceded by much related work, for example Poincaré's analysis of the Dirichlet problem and associated eigenvalues (1890–6). One could maintain that the subject started with the seminal work of Fourier on the solution of the heat equation using series expansions in sines and cosines, which was published by the Académie Française in 1822. Fourier submitted this work in 1807, during the Napoleonic era, and an account of his misfortunes during the fifteen year period before publication is given by Körner (1988). I have included the names and dates associated with a few of the key ideas in the text; a much more comprehensive account may be found in Dieudonné (1981).

Much of the subject matter in the book is confined not only to linear differential operators but even to second order elliptic differential operators. One justification for concentrating on this topic is that many of the equations which have proved important in the physical sciences and engineering over the last century involve operators of this type.

Most important among these is non-relativistic quantum theory, which is based upon the spectral analysis of Schrödinger operators. Applications of second order elliptic operators to geometry and stochastic analysis are also now of great importance. The constraints of time have forced me to omit any account of the wave equation, which certainly has an importance equal to what is included; it seems to me that this theory is best presented within the context of pseudo-differential operator theory, another vast subject. The theory of higher order elliptic operators extends the second order theory in an obvious sense, but loses the close connection which the second order theory has with Brownian motion. Without going into details, I can assure the student that the techniques presented in this book are of central importance to the higher order theory. Non-linear differential equations present a new order of variety and complication, but there also the linear theory is of importance. The KdV equation has solutions expressed in terms of properties of an associated family of linear problems. In many other non-linear problems the proof of local existence theorems depends upon the use of linear theory for operators with very weak assumptions on the coefficients.

Spectral theory is an extremely rich field which has been studied by many qualitative and quantitative techniques – for example Sturm–Liouville theory, separation of variables, Fourier and Laplace transforms, perturbation theory, eigenfunction expansions, variational methods, microlocal analysis, stochastic analysis and numerical methods including finite elements. Whether or not a student is going on to study one of these developments in detail, he or she should have an opportunity to see something of the underlying subject.

The book differs from other introductory texts on elliptic differential operators in that it does not assume that the operators have smooth coefficients, and does not depend upon a heavy use of embedding properties of Sobolev spaces. There is also more concern than in other texts at this level to establish relations between upper and lower bounds on the spectrum and quantitative assumptions about the regions in which the operators act. Many of the theorems are stated and proved in less than maximal generality, in order to make the essential ideas more accessible to students.

The prerequisites for reading the book are a knowledge of some Hilbert and Banach space theory, a course on Lebesgue integration and measure theory, and a little familiarity with Fourier transforms. I assume that the reader is aware of the elements of spectral theory for a bounded linear operator on an abstract Banach space. Chapter 2 presents an entirely new proof of the spectral theorem for unbounded self-adjoint operators. Although this proof is due to the author (Davies, 1993), it originates from a formula of Helffer and Sjöstrand which has had many important applications to spectral and scattering theory over the last few years. The proof is particularly straightforward and direct, and has the further advantage of using techniques which are of value elsewhere in the theory of differential equations. Those who already know a proof of the spectral theorem are not, however, disadvantaged.

The core chapters of the book are numbers 1, 2 and 4, all of which deal with abstract spectral theory; the remaining chapters are to some extent independent of each other, except that Chapter 8, on Schrödinger operators, depends heavily upon the treatment of constant coefficient operators using methods of Fourier analysis in Chapter 3. Most of the subject matter can be covered within a one year course by mathematics students in their fourth or fifth year at University. It is also possible to base a shorter course on the book, by taking Theorems 2.5.1, 4.5.2 and 6.2.3 as unproved statements of fact, and then following up their consequences. While almost all of the material has been known for a considerable time, I have presented it in a form which is related to my own research interests, concentrating particularly on Hilbert space techniques and the variational method. This method has been used for many decades, and has acquired extra strength in the last fifteen years with the increased development of the theory of quadratic forms.

It is possible to regard the present book as the first volume of a three-volume series by the author, the others being *One-Parameter Semigroups* (1980) and *Heat Kernels and Spectral Theory* (1989).

I would like to conclude by thanking the many people who have stimulated my interest in spectral theory over the last twenty years. I would also like to thank my students A. Arnal, O. Nicholas, M. Owen and P. Oleche for their constructive criticisms of the manuscript, and also for their help in eliminating a large number of minor errors. Needless to say I take full responsibility for those which remain.

Kings College, London E. Brian Davies
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