Chapter I

Algebras and modules

We introduce here the notations and terminology we use on algebras and modules, and we briefly recall some of the basic facts from module theory. Examples of algebras, modules, and functors are presented. We introduce the notions of the (Jacobson) radical of an algebra and of a module; the notions of semisimple module, projective cover, injective envelope, the socle and the top of a module, local algebra, and primitive idempotent. We also collect basic facts from the module theory of finite dimensional K-algebras. In this chapter we present complete proofs of most of the results, except for a few classical theorems. In these cases the reader is referred to the following textbooks on this subject [2], [6], [49], [61], [131], and [165].

Throughout, we freely use the basic notation and facts on categories and functors introduced in the Appendix.

The reader interested mainly in linear representations of quivers and path algebras or familiar with elementary facts on rings and modules can skip this chapter and begin with Chapter II.

For the sake of simplicity of presentation, we always suppose that K is an algebraically closed field and that an algebra means a finite dimensional K-algebra, unless otherwise specified.

I.1 Algebras

By a **ring**, we mean a triple $(A, +, \cdot)$ consisting of a set A, two binary operations: addition $+ : A \times A \to A$, $(a, b) \mapsto a + b$; multiplication $\cdot : A \times A \to A$, $(a, b) \mapsto ab$, such that $(A, +)$ is an abelian group, with zero element $0 \in A$, and the following conditions are satisfied:

(i) $(ab)c = a(bc)$,

(ii) $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$

for all $a, b, c \in A$. In other words, the multiplication is associative and both left and right distributive over the addition. A ring A is **commutative** if $ab = ba$ for all $a, b \in A$.

We only consider rings such that there is an element $1 \in A$ where $1 \neq 0$ and $1a = a1 = a$ for all $a \in A$. Such an element is unique with respect to this property; we call it the **identity** of the ring A. In this case the ring is a quadruple $(A, +, \cdot, 1)$. Throughout, we identify the ring $(A, +, \cdot, 1)$ with its underlying set A.

A ring K is a **skew field** (or division ring) if every nonzero element a in K is invertible, that is, there exists $b \in K$ such that $ab = 1$ and $ba = 1$. A skew field K is said to be a **field** if K is commutative.

A field K is **algebraically closed** if any nonconstant polynomial $h(t)$ in one indeterminate t with coefficients in K has a root in K.

If A and B are rings, a map $f : A \to B$ is called a **ring homomorphism** if $f(a + b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$ for all $a, b \in A$. If, in addition, A and B are rings with identity elements we assume that the ring homomorphism f preserves the identities, that is, that $f(1) = 1$.

Let K be a field. A K-**algebra** is a ring A with an identity element (denoted by 1) such that A has a K-vector space structure compatible with the multiplication of the ring, that is, such that

$$\lambda(ab) = (a\lambda)b = a(\lambda b) = (ab)\lambda$$

for all $\lambda \in K$ and all $a, b \in A$. A K-algebra A is said to be **finite dimensional** if the dimension $\dim_K A$ of the K-vector space A is finite.

A K-vector subspace B of a K-algebra A is a K-**subalgebra** of A if the identity of A belongs to B and $bb' \in B$ for all $b, b' \in B$. A K-vector subspace I of a K-algebra A is a **right ideal** of A (or **left ideal** of A) if $xa \in I$ (or $ax \in I$, respectively) for all $x \in I$ and $a \in A$. A two-sided ideal of A (or simply an ideal of A) is a K-vector subspace I of A that is both a left ideal and a right ideal of A.

It is easy to see that if I is a two-sided ideal of a K-algebra A, then the quotient K-vector space A/I has a unique K-algebra structure such that the canonical surjective linear map $\pi : A \to A/I$, $a \mapsto \overline{a} = a + I$, becomes a K-algebra homomorphism.

If I is a two-sided ideal of A and $m \geq 1$ is an integer, we denote by I^m the two-sided ideal of A generated by all elements $x_1x_2 \ldots x_m$, where $x_1, x_2, \ldots, x_m \in I$, that is, I^m consists of all finite sums of elements of the form $x_1x_2 \ldots x_m$, where $x_1, x_2, \ldots, x_m \in I$. We set $I^0 = A$. The ideal I is said to be **nilpotent** if $I^m = 0$ for some $m \geq 1$.

If A and B are K-algebras, then a ring homomorphism $f : A \to B$ is called a K-**algebra homomorphism** if f is a K-linear map. Two K-algebras A and B are called isomorphic if there is a K-algebra isomorphism $f : A \to B$, that is, a bijective K-algebra homomorphism. In this case we write $A \cong B$.

Throughout this book, K denotes an algebraically closed field.

1.1. Examples. (a) The ring $K[t]$ of all polynomials in the indeterminate t with coefficients in K and the ring $K[t_1, \ldots, t_n]$ of all polynomials in commuting indeterminates $t_1, \ldots, t_n$ with coefficients in K are infinite dimensional K-algebras.

(b) If A is a K-algebra and $n \in \mathbb{N}$, then the set $\mathbb{M}_n(A)$ of all $n \times n$ square matrices with coefficients in A is a K-algebra with respect to the usual matrix addition and multiplication. The identity of $\mathbb{M}_n(A)$ is the matrix $E = \operatorname{diag}(1, \ldots, 1) \in \mathbb{M}_n(A)$ with 1 on the main diagonal and zeros elsewhere. In particular $\mathbb{M}_n(K)$ is a K-algebra of dimension n^2. A K-basis of $\mathbb{M}_n(K)$ is the set of matrices e_{ij}, $1 \leq i, j \leq n$, where e_{ij} has the coefficient 1 in the position (i, j) and the coefficient 0 elsewhere.

(c) The subset

$$\mathbb{T}_n(K) = \begin{bmatrix} K & 0 & \ldots & 0 \\ K & K & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ K & K & \ldots & K \end{bmatrix}$$

of $\mathbb{M}_n(K)$ consisting of all triangular matrices $[a_{ij}]$ in $\mathbb{M}_n(K)$ with zeros over the main diagonal is a K-subalgebra of $\mathbb{M}_n(K)$. If $n = 3$ then the subset

$$A = \begin{bmatrix} K & 0 & 0 \\ 0 & K & 0 \\ K & K & K \end{bmatrix}$$

of $\mathbb{M}_3(K)$ consisting of all lower triangular matrices $\lambda = [\lambda_{ij}] \in \mathbb{T}_3(K)$ with $\lambda_{21} = 0$ is a K-subalgebra of $\mathbb{M}_3(K)$, and also of $\mathbb{T}_3(K)$.

(d) Suppose that $(I; \preceq)$ is a finite **poset** (partially ordered set), where $I = \{a_1, \ldots, a_n\}$ and $\preceq$ is a partial order relation on I. The subset

$$KI = \{\, \lambda = [\lambda_{ij}] \in \mathbb{M}_n(K);\ \lambda_{st} = 0 \text{ if } a_s \not\preceq a_t \}$$

of $\mathbb{M}_n(K)$ consisting of all matrices $\lambda = [\lambda_{ij}]$ such that $\lambda_{ij} = 0$ if the relation $a_i \preceq a_j$ does not hold in I is a K-subalgebra of $\mathbb{M}_n(K)$. We call KI the **incidence algebra** of the poset $(I; \preceq)$ with coefficients in K. The matrices $\{e_{ij}\}$ with $a_i \preceq a_j$ form a basis of the K-vector space KI.

Without loss of generality, we may suppose that $I = \{1, \ldots, n\}$ and that $i \preceq j$ implies that $i \geq j$ in the natural order. This can easily be done by a suitable renumbering of the elements in I. In this case, KI takes the form of the lower triangular matrix algebra

$$KI = \begin{bmatrix} K & 0 & \ldots & 0 \\ K_{21} & K & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ K_{n1} & K_{n2} & \ldots & K \end{bmatrix},$$

where $K_{ij} = K$ if $i \preceq j$ and $K_{ij} = 0$ otherwise. For example, if $(I; \preceq)$ is the poset $\{1 \succ 2 \succ 3 \succ \cdots \succ n\}$ then the algebra KI is isomorphic to the algebra $\mathbb{T}_n(K)$ in Example 1.1 (c). If $(I; \preceq)$ is the poset $\{1 \succ 3 \prec 2\}$ then the incidence algebra KI is isomorphic to the five-dimensional algebra A in Example 1.1 (c). If the poset $(I;\preceq)$ is given by $I=\{1,2,3,4\}$ and the relations $\{3\succ 4\prec 2\prec 1\succ 3\}$ then

$$KI=\begin{bmatrix} K & 0 & 0 & 0\\ K & K & 0 & 0\\ K & 0 & K & 0\\ K & K & K & K\end{bmatrix}.$$

(e) The associative ring $K\langle t_1,t_2\rangle$ of all polynomials in two noncommuting indeterminates t_1 and t_2 with coefficients in K is an infinite dimensional K-algebra. Note that, if I is the two-sided ideal in $K\langle t_1,t_2\rangle$ generated by the element $t_1t_2-t_2t_1$, then the K-algebra $K\langle t_1,t_2\rangle/I$ is isomorphic to $K[t_1,t_2]$.

(f) Let $(G,\cdot)$ be a finite group with identity element e and let A be a K-algebra. The **group algebra** of G with coefficients in A is the K-vector space AG consisting of all the formal sums $\sum_{g\in G} g\,\lambda_g$, where $\lambda_g\in A$ and $g\in G$, with the multiplication defined by the formula

$$\Big(\sum_{g\in G} g\,\lambda_g\Big)\cdot\Big(\sum_{h\in G} h\,\mu_h\Big)=\sum_{f=gh\in G} f\lambda_g\mu_h.$$

Then AG is a K-algebra of dimension $|G|\cdot\dim_K A$ (here $|G|$ denotes the order of G) and the element $e=e1$ is the identity of AG. If $A=K$, then the elements $g\in G$ form a basis of KG over K.

For example, if G is a cyclic group of order m, then $KG\cong K[t]/(t^m-1)$.

(g) Assume that A_1 and A_2 are K-algebras. The **product of the algebras** A_1 and A_2 is the algebra $A=A_1\times A_2$ with the addition and the multiplication given by the formulas $(a_1,a_2)+(b_1,b_2)=(a_1+b_1,a_2+b_2)$ and $(a_1,a_2)(b_1,b_2)=(a_1b_1,a_2b_2)$, where $a_1,b_1\in A_1$ and $a_2,b_2\in A_2$. The identity of A is the element $1=(1,1)=e_1+e_2\in A_1\times A_2$, where $e_1=(1,0)$ and $e_2=(0,1)$.

(h) For any K-algebra A we define the **opposite algebra** $A^{\rm op}$ of A to be the K-algebra whose underlying set and vector space structure are just those of A, but the multiplication $*$ in $A^{\rm op}$ is defined by formula $a*b=ba$.

1.2. Definition. The (Jacobson) **radical** $\operatorname{rad} A$ of a K-algebra A is the intersection of all the maximal right ideals in A.

It follows from (1.3) that $\operatorname{rad} A$ is the intersection of all the maximal left ideals in A. In particular, $\operatorname{rad} A$ is a two-sided ideal.

1.3. Lemma. *Let A be a K-algebra and let $a\in A$. The following conditions are equivalent:*

(a) *$a\in\operatorname{rad} A$;*

(a′) *a belongs to the intersection of all maximal left ideals of A;*
(b) *for any $b \in A$, the element $1 - ab$ has a two-sided inverse;*
(b′) *for any $b \in A$, the element $1 - ab$ has a right inverse;*
(c) *for any $b \in A$, the element $1 - ba$ has a two-sided inverse;*
(c′) *for any $b \in A$, the element $1 - ba$ has a left inverse.*

Proof. (a) implies (b′). Let $b \in A$ and assume to the contrary that $1 - ab$ has no right inverse. Then there exists a maximal right ideal I of A such that $1 - ab \in I$. Because $a \in \operatorname{rad} A \subseteq I$, $ab \in I$ and $1 \in I$; this is a contradiction. This shows that $1 - ab$ has a right inverse.

(b′) implies (a). Assume to the contrary that $a \notin \operatorname{rad} A$ and let I be a maximal right ideal of A such that $a \notin I$. Then $A = I + aA$ and therefore there exist $x \in I$ and $b \in A$ such that $1 = x + ab$. It follows that $x = 1 - ab \in I$ has no right inverse, contrary to our assumption. The equivalence of (a′) and (c′) can be proved in a similar way.

The equivalence of (b) and (c) is a consequence of the following two simple implications:

(i) If $(1 - cd)x = 1$, then $(1 - dc)(1 + dxc) = 1$.
(ii) If $y(1 - cd) = 1$, then $(1 + dyc)(1 - dc) = 1$.

(b′) implies (b). Fix an element $b \in A$. By (b′), there exists an element $c \in A$ such that $(1 - ab)c = 1$. Hence $c = 1 - a(-bc)$ and, according to (b′), there exists $d \in A$ such that $1 = cd = d + abcd = d + ab$. It follows that $d = 1 - ab$, c is the left inverse of $1 - ab$ and (b) follows. That (c′) implies (c) follows in a similar way. Because (b) implies (b′) and (c) implies (c′) obviously, the lemma is proved. □

1.4. Corollary. *Let $\operatorname{rad} A$ be the radical of an algebra A.*
(a) *$\operatorname{rad} A$ is the intersection of all the maximal left ideals of A.*
(b) *$\operatorname{rad} A$ is a two-sided ideal and $\operatorname{rad}(A/\operatorname{rad} A) = 0$.*
(c) *If I is a two-sided nilpotent ideal of A, then $I \subseteq \operatorname{rad} A$. If, in addition, the algebra A/I is isomorphic to a product $K \times \cdots \times K$ of copies of K, then $I = \operatorname{rad} A$.*

Proof. The statements (a) and (b) easily follow from (1.3).

(c) Assume that $I^m = 0$ for some $m > 0$. Let $x \in I$ and let a be an element of A. Then $ax \in I$ and therefore $(ax)^r = 0$ for some $r > 0$. It follows that the equality $(1 + ax + (ax)^2 + \cdots + (ax)^{r-1})(1 - ax) = 1$ holds for any element $a \in A$, and, according to (1.3), the element x belongs to $\operatorname{rad} A$. Consequently, $I \subseteq \operatorname{rad} A$. To prove the reverse inclusion, assume that the algebra A/I is isomorphic to a product of copies of K. It follows that $\operatorname{rad}(A/I) = 0$. Next, the canonical surjective algebra homomorphism $\pi : A \to A/I$ carries $\operatorname{rad} A$ to $\operatorname{rad}(A/I) = 0$. Indeed, if $a \in \operatorname{rad} A$ and $\pi(b) = b + I$, with $b \in A$, is any element of A/I then, by (1.3), $1 - ba$ is invertible in A and therefore the element $\pi(1-ba) = 1-\pi(b)\pi(a)$ is invertible in A/I; thus $\pi(a) \in \operatorname{rad} A/I = 0$, by (1.3). This yields $\operatorname{rad} A \subseteq \operatorname{Ker} \pi = I$ and finishes the proof. □

1.5. Examples. (a) Let $s_1, \ldots, s_n$ be positive integers and let $A = K[t_1, \ldots, t_n]/(t_1^{s_1}, \ldots, t_n^{s_n})$. Because the ideal $I = (\overline{t}_1, \ldots, \overline{t}_n)$ of A generated by the cosets $\overline{t}_1, \ldots, \overline{t}_n$ of the indeterminates $t_1, \ldots, t_n$ modulo the ideal $(t_1^{s_1}, \ldots, t_n^{s_n})$ is nilpotent, then (1.4) yields $I \subseteq \operatorname{rad} A$. On the other hand, there is a K-algebra isomorphism $A/I \cong K$. It follows that I is a maximal ideal and therefore $\operatorname{rad} A = I$.

(b) Let I be a finite poset and $A = KI$ be its incidence K-algebra viewed, as in (1.1)(d), as a subalgebra of the full matrix algebra $\mathbb{M}_n(K)$. Then $\operatorname{rad} A$ is the set U of all matrices $\lambda = [\lambda_{ij}] \in KI$ with $\lambda_{ii} = 0$ for $i = 1, 2, \ldots, n$, and the algebra $A/\operatorname{rad} A$ is isomorphic to the product $K \times \cdots \times K$ of n copies of K. Indeed, we note that the set U is clearly a two-sided ideal of KI, it is easily seen that $U^n = 0$ and finally the algebra A/U is isomorphic to the product of n copies of K, thus (1.4)(c) applies.

(c) By applying the preceding arguments, one also shows that the radical $\operatorname{rad} A$ of the lower triangular matrix algebra $A = \mathbb{T}_n(K)$ of (1.1)(c) consists of all matrices in A with zeros on the main diagonal. It follows that $(\operatorname{rad} A)^n = 0$.

In the study of modules over finite dimensional K-algebras over an algebraically closed field K an important rôle is played by the following theorem, known as the Wedderburn–Malcev theorem.

1.6. Theorem. *Let A be a finite dimensional K-algebra. If the field K is algebraically closed, then there exists a K-subalgebra B of A such that there is a K-vector space decomposition $A = B \oplus \operatorname{rad} A$ and the restriction of the canonical surjective algebra homomorphism $\pi : A \to A/\operatorname{rad} A$ to B is a K-algebra isomorphism.*

Proof. See [61, section VI.2] and [131, section 11.6]. □

I.2 Modules

2.1. Definition. Let A be a K-algebra. A **right A-module** (or a right module over A) is a pair $(M, \cdot)$, where M is a K-vector space and $\cdot : M \times A \to M$, $(m, a) \mapsto ma$, is a binary operation satisfying the following conditions:

(a) $(x + y)a = xa + ya$;
(b) $x(a + b) = xa + xb$;
(c) $x(ab) = (xa)b$;
(d) $x1 = x$;

(e) $(x\lambda)a = x(a\lambda) = (xa)\lambda$

for all $x, y \in M$, $a, b \in A$ and $\lambda \in K$.

The definition of a left A-module is analogous. Throughout, we write M or M_A instead of $(M, \cdot)$. We write A_A and ${}_AA$ whenever we view the algebra A as a right or left A-module, respectively.

A module M is said to be **finite dimensional** if the dimension $\dim_K M$ of the underlying K-vector space of M is finite.

A K-subspace M' of a right A-module M is said to be an A-**submodule** of M if $ma \in M'$ for all $m \in M'$ and all $a \in A$. In this case the K-vector space M/M' has a natural A-module structure such that the canonical epimorphism $\pi : M \to M/M'$ is an A-module homomorphism.

Let M be a right A-module and let I be a right ideal of A. It is easy to see that the set MI consisting of all sums $m_1a_1 + \ldots + m_sa_s$, where $s \geq 1$, $m_1, \ldots, m_s \in M$ and $a_1, \ldots, a_s \in I$, is a submodule of M.

A right A-module M is said to be **generated** by the elements $m_1, \ldots, m_s$ of M if any element $m \in M$ has the form $m = m_1a_1 + \cdots + m_sa_s$ for some $a_1, \ldots, a_s \in A$. In this case, we write $M = m_1A + \ldots + m_sA$. A module M is said to be **finitely generated** if it is generated by a finite subset of elements of M.

Let $M_1, \ldots, M_s$ be submodules of a right A-module M. We define $M_1 + \ \ldots\ + M_s$ to be the submodule of M consisting of all sums $m_1 + \cdots + m_s$, where $m_1 \in M_1, \cdots, m_s \in M_s$, and we call it the submodule generated by $M_1, \ldots, M_s$, or the sum of $M_1, \ldots, M_s$.

Note that a right module M over a finite dimensional K-algebra A is finitely generated if and only if M is finite dimensional. Indeed, if $x_1, \ldots, x_m$ is a K-basis of M, then it is obviously a set of A-generators of M. Conversely, if the A-module M is generated by the elements $m_1, \ldots, m_n$ over A and $\xi_1, \ldots \xi_s$ is a K-basis of A then the set $\{m_j\xi_i; j = 1, \ldots, n, i = 1, \ldots, s\}$ generates the K-vector space M.

Throughout, we frequently use the following lemma, known as Nakayama's lemma.

2.2. Lemma. *Let A be a K-algebra, M be a finitely generated right A-module, and $I \subseteq \operatorname{rad} A$ be a two-sided ideal of A. If $MI = M$, then $M = 0$.*

Proof. Suppose that $M = MI$ and $M = m_1A + \cdots + m_sA$, that is, M is generated by the elements $m_1, \ldots, m_s$. We proceed by induction on s. If $s = 1$, then the equality $m_1A = m_1I$ implies that $m_1 = m_1x_1$ for some $x_1 \in I$. Hence $m_1(1 - x_1) = 0$ and therefore $m_1 = 0$, because $1 - x_1$ is invertible. Consequently $M = 0$, as required.

Assume that $s \geq 2$. The equality $M = MI$ implies that there are elements $x_1, \ldots, x_s \in I$ such that $m_1 = m_1x_1 + m_2x_2 + \cdots + m_sx_s$. Hence $m_1(1 - x_1) = m_2x_2 + \cdots + m_sx_s$ and therefore $m_1 \in m_2A + \cdots + m_sA$ because $1 - x_1$ is invertible. This shows that $M = m_2A + \cdots + m_sA$ and the inductive hypothesis yields $M = 0$. □

2.3. Corollary. *If A is a finite dimensional K-algebra, then* $\operatorname{rad} A$ *is nilpotent.*

Proof. Because $\dim_K A < \infty$, the chain

$$A \supseteq \operatorname{rad} A \supseteq (\operatorname{rad} A)^2 \supseteq \cdots \supseteq (\operatorname{rad} A)^m \supseteq (\operatorname{rad} A)^{m+1} \supseteq \cdots$$

becomes stationary. It follows that $(\operatorname{rad} A)^m = (\operatorname{rad} A)^m \operatorname{rad} A$ for some m, and Nakayama's lemma (2.2) yields $(\operatorname{rad} A)^m = 0$. □

Let M and N be right A-modules. A K-linear map $h : M \to N$ is said to be an **A-module homomorphism** (or simply an A-homomorphism) if $h(ma) = h(m)a$ for all $m \in M$ and $a \in A$. An A-module homomorphism $h : M \to N$ is said to be a **monomorphism** (or an **epimorphism**) if it is injective (or surjective, respectively). A bijective A-module homomorphism is called an **isomorphism**. The right A-modules M and N are said to be **isomorphic** if there exists an A-module isomorphism $h : M \to N$. In this case, we write $M \cong N$. An A-module homomorphism $h : M \to M$ is said to be an **endomorphism** of M.

The set $\operatorname{Hom}_A(M, N)$ of all A-module homomorphisms from M to N is a K-vector space with respect to the scalar multiplication $(f, \lambda) \mapsto f\lambda$ given by $(f\lambda)(m) = f(m\lambda)$ for $f \in \operatorname{Hom}_A(M, N)$, $\lambda \in K$ and $m \in M$. If the modules M and N are finite dimensional, then the K-vector space $\operatorname{Hom}_A(M, N)$ is finite dimensional. The K-vector space

$$\operatorname{End} M = \operatorname{Hom}_A(M, M)$$

of all A-module endomorphisms of any right A-module M is an associative K-algebra with respect to the composition of maps. The identity map 1_M on M is the identity of $\operatorname{End} M$.

It is easy to check that for any triple L, M, N of right A-modules the composition mapping $\cdot : \operatorname{Hom}_A(M, N) \times \operatorname{Hom}_A(L, M) \longrightarrow \operatorname{Hom}_A(L, N)$, $(h, g) \mapsto hg$, is K-bilinear.

It is clear that the **kernel** $\operatorname{Ker} h = \{m \in M \mid h(m) = 0\}$, the **image** $\operatorname{Im} h = \{h(m) \mid m \in M\}$, and the **cokernel** $\operatorname{Coker} h = N/\operatorname{Im} h$ of an A-module homomorphism $h : M \to N$ have natural A-module structures.

The **direct sum** of the right A-modules $M_1, \ldots, M_s$ is defined to be the K-vector space direct sum $M_1 \oplus \cdots \oplus M_s$ equipped with an A-module structure defined by $(m_1, \ldots, m_s)a = (m_1a, \ldots, m_sa)$ for $m_1 \in M_1, \ldots, m_s \in M_s$

and $a \in A$. We set

$$M^s = M \oplus \cdots \oplus M, \quad (s \text{ copies}).$$

A right A-module M is said to be **indecomposable** if M is nonzero and M has no direct sum decomposition $M \cong N \oplus L$, where L and N are nonzero A-modules.

We denote by $\operatorname{Mod} A$ the abelian category of all right A-modules, that is, the category whose objects are right A-modules, the morphisms are A-module homomorphisms, and the composition of morphisms is the usual composition of maps. The reader is referred to Sections 1 and 2 of the Appendix for basic facts on categories and functors. Throughout, we freely use the notation introduced there.

We note that any left A-module can be viewed as a right A^{op}-module and conversely. Therefore, throughout the text, the category $\operatorname{Mod} A^{\mathrm{op}}$ is identified with the category of all left A-modules.

We denote by $\operatorname{mod} A$ the full subcategory of $\operatorname{Mod} A$ whose objects are the finitely generated modules. It follows that if A is a finite dimensional K-algebra, then all modules in $\operatorname{mod} A$ are finite dimensional.

An important idea in the study of A-modules is to view them as sets of K-vector spaces connected by K-linear maps. This is illustrated by the following three examples.

2.4. Example. Let A be the lower triangular matrix K-subalgebra

$$A = \begin{bmatrix} K & 0 \\ K & K \end{bmatrix}$$

of the matrix algebra $\mathbb{M}_2(K)$. We note that the matrices $e_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $e_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $e_{21} = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ form a K-basis of A over K, $1_R = e_1 + e_2$, and $e_1 e_2 = e_2 e_1 = 0$.

It follows that every module X in $\operatorname{mod} A$, viewed as a K-vector space, has a direct sum decomposition $X = X_1 \oplus X_2$, where X_1, X_2 are the vector spaces Xe_1, Xe_2 over K. Note that given $a = \left(\begin{smallmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{smallmatrix}\right) \in A$ and $x = (x_1, x_2) \in X$ with $x_1 \in X_1$ and $x_2 \in X_2$ we have

$$xa = (x_1 a_{11} + x_2 a_{21}, x_2 a_{22}) = (x_1 a_{11} + f_X(x_2) a_{21}, x_2 a_{22}),$$

where $f_X : X_2 \to X_1$ is the K-linear map given by the formula $f_X(x_2) = x_2 e_{21} = x_2 e_{21} e_{11}$. It follows that X, viewed as a right A-module, can be identified with the triple $(X_1 \xleftarrow{f_X} X_2)$. Moreover, any A-module homomorphism $h : X \to Y$ can be identified with the pair (h_1, h_2) of K-linear maps $h_1 : X_1 \to Y_1$, $h_2 : X_2 \to Y_2$ that are the restrictions of h to, respectively, X_1 and X_2. These satisfy the equation $h_1 f_X = f_Y h_2$.

The converse correspondence to $X \mapsto (X_1 \xleftarrow{f_X} X_2)$ is defined by associating to any triple $(X_1 \xleftarrow{f} X_2)$ with K-vector spaces X_1, X_2 and

$f \in \mathrm{Hom}_K(X_2, X_1)$, the K-vector space $X = X_1 \oplus X_2$ endowed with the right action $\cdot : X \times A \to X$ of A on X defined by the formula $(x_1, x_2)\left(\begin{smallmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{smallmatrix}\right) = (x_1 a_{11} + f(x_2)a_{21}, x_2 a_{22})$, where $x_1 \in X_1$, $x_2 \in X_2$, and $\left(\begin{smallmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{smallmatrix}\right) \in A$.

2.5. Example. Let A be the **Kronecker algebra**

$$A = \begin{bmatrix} K & 0 \\ K^2 & K \end{bmatrix}$$

whose elements are 2×2 matrices of the form $\left(\begin{smallmatrix} \lambda & 0 \\ (u_1, u_2) & \mu \end{smallmatrix}\right)$ with $\lambda, \mu \in K$, $(u_1, u_2) \in K^2$, and the multiplication in A is defined by the formula

$$\begin{pmatrix} d & 0 \\ (u_1,u_2) & c \end{pmatrix} \begin{pmatrix} f & 0 \\ (v_1,v_2) & e \end{pmatrix} = \begin{pmatrix} df & 0 \\ (u_1 f + v_1 c, u_2 f + v_2 c) & ce \end{pmatrix}.$$

Finite dimensional right A-modules are called **Kronecker modules**. Every such A-module X can be identified with a quadruple

$$\Big(X_1 \overset{\varphi_1}{\underset{\varphi_2}{\longleftleftarrows}} X_2 \Big),$$

where X_1, X_2 are the K-vector spaces Xe_1, Xe_2, respectively, $e_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $e_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, φ_1, φ_2 are the K-linear maps defined by the formulas

$$\varphi_1(x) = x \cdot \left(\begin{smallmatrix} 0 & 0 \\ \xi_1 & 0 \end{smallmatrix}\right) = x \cdot \left(\begin{smallmatrix} 0 & 0 \\ \xi_1 & 0 \end{smallmatrix}\right) \cdot e_1, \quad \varphi_2(x) = x \cdot \left(\begin{smallmatrix} 0 & 0 \\ \xi_2 & 0 \end{smallmatrix}\right) = x \cdot \left(\begin{smallmatrix} 0 & 0 \\ \xi_2 & 0 \end{smallmatrix}\right) \cdot e_1,$$

for $x \in X_2$, where $\xi_1 = (1, 0)$ and $\xi_2 = (0, 1)$ are the standard basis vectors of K^2. Any A-module homomorphism $c : X' \to X$ can be identified with a pair (c_1, c_2) of K-linear maps $c_1 : X'_1 \to X_1$ and $c_2 : X'_2 \to X_2$ such that $c_1 \varphi'_1 = \varphi_1 c_2$ and $c_1 \varphi'_2 = \varphi_2 c_2$.

The converse correspondence to $X \mapsto (X_1 \overset{\varphi_1}{\underset{\varphi_2}{\longleftleftarrows}} X_2)$ is defined by associating to any quadruple $(X_1 \overset{\varphi_1}{\underset{\varphi_2}{\longleftleftarrows}} X_2)$ with finite dimensional K-vector spaces X_1, X_2 and $\varphi_1, \varphi_2 \in \mathrm{Hom}_K(X_2, X_1)$, the K-vector space $X = X_1 \oplus X_2$ endowed with the right action $\cdot : X \times A \to X$ of A on X defined by the formula

$$(x_1, x_2)\begin{pmatrix} \lambda & 0 \\ (u_1,u_2) & \mu \end{pmatrix} = (x_1 \lambda + \varphi_1(x_2)u_1 + \varphi_1(x_2)u_2,\ x_2 \mu),$$

where $x_1 \in X_1$, $x_2 \in X_2$ and $\begin{pmatrix} \lambda & 0 \\ (u_1,u_2) & \mu \end{pmatrix} \in A$.

It follows that the category of Kronecker modules is equivalent to the category of pairs $[\Phi_1, \Phi_2]$ of matrices Φ_1, Φ_2 over K of the same size, where the map from $[\Phi'_1, \Phi'_2]$ to $[\Phi_1, \Phi_2]$ is a pair (C_1, C_2) of matrices with coefficients in K such that $C_1 \Phi'_1 = \Phi_1 C_2$ and $C_1 \Phi'_2 = \Phi_2 C_2$.

2.6. Example. Let $K[t]$ be the K-algebra of all polynomials in the indeterminate t with coefficients in K. Note that every module V in $\mathrm{Mod}\, K[t]$