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Elements of the Representation Theory of Associative Algebras

Volume 1 Techniques of Representation Theory

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Contents

Introduction

The idea of representing a complex mathematical object by a simpler one is as old as mathematics itself. It is particularly useful in classification problems. For instance, a single linear transformation on a finite dimensional vector space is very adequately characterised by its reduction to its rational or its Jordan canonical form. It is now generally accepted that the representation theory of associative algebras traces its origin to Hamilton's description of the complex numbers by pairs of real numbers. During the 1930s, E. Noether gave to the theory its modern setting by interpreting representations as modules. That allowed the arsenal of techniques developed for the study of semisimple algebras as well as the language and machinery of homological algebra and category theory to be applied to representation theory. Using these, the theory grew rapidly over the past thirty years.

Nowadays, studying the representations of an algebra (which we always assume to be finite dimensional over an algebraically closed field, associative, and with an identity) is understood as involving the classification of the (finitely generated) indecomposable modules over that algebra and the homomorphisms between them. The rapid growth of the theory and the extent of the published original literature became major obstacles for the beginners seeking to make their way into this area.

We are writing this textbook with these considerations in mind: It is therefore primarily addressed to graduate students starting research in the representation theory of algebras. It should also, we hope, be of interest to mathematicians working in other fields.

At the origin of the present developments of the theory is the almost simultaneous introduction and use on the one hand of quiver-theoretical techniques by P. Gabriel and his school and, on the other hand, of the theory of almost split sequences by M. Auslander, I. Reiten, and their students. An essential rôle in the theory is also played by integral quadratic forms. Our approach in this book consists in developing these theories on an equal footing, using their interplay to obtain our main results. Our strong belief is that this combination is best at yielding both concrete illustrations of the concepts and the theorems and an easier computation of actual examples. We have thus taken particular care in introducing in the text as many as possible of the latter and have included a large number of workable exercises.

With these purposes in mind, we divide our material into two parts.

The first volume serves as a general introduction to some of the techniques most commonly used in representation theory. We start by showing in Chapters II and III how one can represent an algebra by a bound quiver and a module by a linear representation of the bound quiver. We then turn in Chapter IV to the Auslander–Reiten theory of almost split sequences, giving various characterisations of these, showing their existence in module categories, and introducing one of our main working tools, the so-called Auslander–Reiten quiver. As a first and easy application of these concepts, we show in Chapter V how one can obtain a complete description of the representation theory of the Nakayama (or generalised uniserial) algebras. We return to theory in Chapter VI, giving an outline of tilting theory, another of our main working tools. A first application of tilting theory is the classification in Chapter VII of those hereditary algebras that are representation-finite (that is, admit only finitely many isomorphism classes of indecomposable modules) by means of the Dynkin diagrams, a result now known as Gabriel's theorem. We then study in Chapter VIII a class of algebras whose representation theory is as "close" as possible to that of hereditary algebras, the class of tilted algebras introduced by D. Happel and C. M. Ringel. Besides the general properties of tilted algebras, we give a very handy criterion, due to S. Liu and A. Skowroński, allowing verification of whether a given algebra is tilted or not. The last chapter in this volume deals with indecomposable modules not lying on an oriented cycle of nonzero nonisomorphisms between indecomposable modules.

Throughout this volume, we essentially use integral quadratic form techniques. We present them here in the spirit of Ringel [145].

The first volume ends with an appendix collecting, for the convenience of the reader, the notations and terminology on categories, functors, and homology and recalling some of the basic facts from category theory and homological algebra needed in the book. In Chapter I, we introduce the notation and terminology we use on algebras and modules, and we briefly recall some of the basic facts from module theory. We introduce the notions of the radical of an algebra and of a module; the notions of semisimple module, projective cover, injective envelope, the socle, and the top of a module, local algebra, primitive idempotent. We also collect basic facts from the module theory of finite dimensional K-algebras.

The reader interested mainly in linear representations of quivers and path algebras or familiar with elementary facts on rings and modules can skip Chapter I.

It is our experience that the contents of the first volume of this book can be covered during one (eight-month) course.

The main aim of the second volume, "Representation-Infinite Tilted Algebras," is to study some interesting classes of representation-infinite algebras A and, in particular, to give a fairly complete description of the representation theory of representation-infinite tilted algebras. If the algebra A is tame hereditary, that is, if the underlying graph of its quiver is a Euclidean diagram, we show explicitly how to compute the regular indecomposable modules over A, and then over any tame concealed algebra.

It was not possible to be encyclopedic in this work. Therefore many important topics from the theory have been left out. Among the most notable omissions are covering techniques, the use of derived categories and partially ordered sets. Some other aspects of the theory presented here are discussed in the books [21], [31], [77], [99], [85], [152], and especially [145].

Throughout this book, the symbols $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$ mean the sets of natural numbers, integers, rational, real, and complex numbers, and $\mathbb{M}_n(K)$ means the set of all square $n \times n$ matrices over K. The cardinality of a set X is denoted by $|X|$.

We take pleasure in thanking all our colleagues and students who helped us with their comments and suggestions. We wish particularly to express our appreciation to Sheila Brenner, Otto Kerner, and Kunio Yamagata for their helpful discussions and suggestions. Particular thanks are due to François Huard and Jessica Lévesque, and Mrs. Jolanta Szelatyńska for her help in preparing a camera-ready copy of the manuscript.