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Introduction to Subfactors

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Dedicated
to
Our wives

Contents

Preface

It must be stated at the outset that this little monograph has no pretensions to being a general all-purpose text in operator algebras. On the contrary, it is an attempt to introduce the potentially interested reader – be it a graduate student or a working mathematician who is not necessarily an operator algebraist – to a selection of topics in the theory of subfactors, this selection being influenced by the authors' tastes and personal viewpoints. For instance, we restrict ourselves to the theory of (usually hyperfinite) II_1 factors and their subfactors (almost always of finite index); thus, factors of type III do not make an appearance beyond the first (introductory) chapter, and the Tomita–Takesaki theorem makes only a cameo appearance in the appendix. It is hoped that such 'simplifications' will help to make the material more accessible to the uninitiated reader.

The aim of this book is to give an introduction to some of the beautiful ideas and results which have been developed, since the inception of the theory of subfactors, by such mathematicians as Adrian Ocneanu and Sorin Popa; an attempt has been made to keep the material as self-contained as possible; in fact, we feel it should be possible to use this monograph as the basis of a two-semester course to second year graduate students with a minimal background in Hilbert space theory.

A remark is in order, as far as the references are concerned; when we state certain standard facts without proof, the reader is often referred to a text in operator algebras; if, in this process, it seems that the text by the second author is cited more often than other texts, that is simply because, a reference for that exact fact already being known to exist at that particular place, it was possible to avoid a search for that fact in other texts.

We now give a brief outline of the contents of this volume for the sake of the possibly interested specialist.

The first chapter begins with a quick introduction to some preliminary facts about von Neumann algebras (Murray–von Neumann classification of factors and introduction to traces); this first section contains no proofs, but most of the sequel – with the notable exception of Popa's theorem on amenable subfactors – is self-contained modulo the unproved facts here. The next section starts with the *GNS* construction and goes on to discuss the *standard form* of a finite von Neumann algebra and, in particular, identifies the commutant of the left-regular representation with the range of the right-regular representation. The chapter continues with a discussion of crossed products by countable groups and concludes with some examples – the left von Neumann algebra of an *ICC* group, factors of the three types coming from crossed products of commutative von Neumann algebras with countable groups acting ergodically and freely, a model of the hyperfinite factor which demonstrates that its fundamental group is the entire positive line, and finally infinite tensor products and the definition of the hyperfinite factor.

The second chapter starts with the classification of (separable) modules over a von Neumann algebra (with separable pre-dual), continues with the definition and some elementary properties of the M-dimension of a (separable) module over a II_1 factor M, and concludes with the definition and some elementary properties of the index of a subfactor of a II_1 factor, the statement of the result on restrictions on index values and a proof of the fact that all index values in the interval $[4, \infty)$ are possible.

The third chapter begins with a section on the fundamental notion of the basic construction; the next section gathers together the basic facts about inclusions of finite-dimensional C^*-algebras (including the fact about the basic construction and 'reflection of Bratteli diagrams' as well as the notion of a Markov trace); the final section introduces the all-important sequence $\{e_n\}$, derives the basic properties of this sequence, and indicates the relation between the theorem on restrictions of index values and the classification of non-negative integral matrices of norm less than 2, as well as an outline of the procedure originally adopted to prove the existence of hyperfinite subfactors of index $4\cos^2\frac{\pi}{n}$.

The fourth chapter is devoted to the principal (or standard) graph invariant of a subfactor. It starts with a discussion of ('bifinite') bimodules over a pair of II_1 factors (contragredients and tensor products); the second section gives the two descriptions (in terms of the sequence of higher relative commutants as well as in terms of the bimodules that occur in the tower of the basic construction) of the principal graph of a subfactor, reduces the result on restriction of index values to the classification of non-negative integral matrices of small norm, and concludes with some examples of principal graphs; the chapter concludes with a discussion of 'Pimsner–Popa bases' and a proof of why the higher relative commutants have an interpretation as intertwiners of bimodules.

The fifth chapter starts with Pimsner and Popa's minimax characterisation of the index of a subfactor, the consequent estimation of the index of a hyperfinite subfactor in terms of an approximating 'ladder' of finite-dimensional C^*-algebras, and introduces the important notion of a commuting square; the second section discusses examples of commuting squares (vertex and spin models, and the braid-group example); the next section is devoted to the relation between commuting squares and the basic construction, and the consequent importance of symmetric or non-degenerate commuting squares (with respect to the Markov trace); the next two sections are devoted to the path-algebra model for a tower of finite-dimensional C^*-algebras, and the reformulation of the commuting square condition in terms of 'biunitarity', respectively; the sixth section is a discussion (without proofs) of the canonical commuting square associated to a subfactor and Popa's theorem on the completeness of this invariant; the final section of this chapter centres around

Ocneanu's compactness theorem and the prescription it provides for computing the higher relative commutants of a subfactor built from an arbitrary initial commuting square.

The sixth and final chapter is devoted to the rich class of examples of subfactors provided by the so-called vertex and spin models. This chapter systematically develops a diagrammatic formulation to discuss the higher relative commutants in these examples, and also shows how to push this diagrammatic formulation through for general commuting squares.

The book concludes with an appendix, which contains some facts used in the text – such as the non-existence of two-sided (algebraic) ideals in finite factors – as well as a computation of the principal and dual graphs of the 'subgroup-subfactor' and the original derivation of the one-variable polynomial invariant of knots.

Finally, the book comes equipped with such customary trappings as a bibliography, some remarks of a bibliographic nature, and one index containing both terms and symbols used.

Acknowledgement

These notes have a two-part genesis. To start with, the first author gave a series of about seven lectures at the centres of the Indian Statistical Institute at New Delhi and Bangalore during the winter of '92–'93; these lectures were 'panoramic' in nature and did not contain too many proofs (because of the obvious time constraints). Later, the second author visited the Research Institute of Mathematical Sciences at Kyoto for nine months during '94–'95, during which period he gave a course of about 25 lectures – which was an expanded version of the earlier abridged version, and which contained proofs of most assertions and was addressed to the graduate students in the audience. These notes were the by-product of those two courses of lectures.

The first (resp., second) author would like to express his gratitude to Professors Kalyan Sinha and V. Radhakrishnan, the Indian Statistical Institute, the Raman Research Institute and the National Board for Higher Mathematics in India (resp., Professor Huzihiro Araki and the Research Institute of Mathematical Sciences, Kyoto University) for the hospitality and the roles they played in making these visits possible.