Chapter 1

Factors

1.1 von Neumann algebras and factors

The fundamental notion is that of a von Neumann algebra. This is, typically, what may be called the 'symmetries of a group'. The precise way of saying this is that a (concrete) *von Neumann algebra* is nothing but a set of the form $M = \pi(G)'$ – where π is a unitary representation of a group G on a Hilbert space $\mathcal{H}$, and S' denotes, for S a subset of $\mathcal{L}(\mathcal{H})$ (the algebra of all bounded operators on $\mathcal{H}$), the commutant of S defined by $S' = \{x' \in \mathcal{L}(\mathcal{H}): x'x = xx' \text{ for all } x \text{ in } S\}$. In other words M is the set of intertwiners of the representation π: thus, $x \in M \Leftrightarrow x\pi(g) = \pi(g)x$ for all g in G.

The usual definition of a von Neumann algebra is: 'a self-adjoint subalgebra of $\mathcal{L}(\mathcal{H})$ satisfying $M = M''$' (where we write M'' for $(M')'$). This is equivalent to the definition we have chosen to give. Reason: if $M = \pi(G)'$, then M is a self-adjoint subalgebra and $M = M''$, since $S' = S'''$ for all $S \subseteq \mathcal{L}(\mathcal{H})$; conversely, if $M = M''$ is a self-adjoint subalgebra, we may set G equal to the unitary group of M' and appeal to the almost obvious fact (cf. [Sun1], Lemma 0.4.7) that G linearly spans M' (so that $G' = (M')'$).

The canonical commutative examples of abstract von Neumann algebras turn out to be $L^\infty(X, \mu)$, while the basic non-commutative example is $\mathcal{L}(\mathcal{H})$.

The fundamental 'double commutant theorem' of von Neumann states that a self-adjoint unital subalgebra M of $\mathcal{L}(\mathcal{H})$ is weakly closed – i.e. $\langle x_n\xi, \eta\rangle \to \langle x\xi, \eta\rangle\ \forall \xi, \eta \in \mathcal{H}, x_n \in M\ \forall n \Rightarrow x \in M$) (if and only if M is σ-weakly closed – see §A.1) if and only if $M = M''$.

The importance of the notion of a von Neumann algebra was recognised in 1936 by Murray and von Neumann (although of course, they called them 'rings of operators') – see [MvN1] – who also quickly realised that the 'building blocks' in the theory of von Neumann algebras were (what they, and people after them, called) *factors*. If $M = \pi(G)'$, then M is a factor precisely when the representation is 'isotypical'. (If G is compact and π is a strongly continous representation, this says that π is a multiple of an irreducible representation;

for general G, this says that any two non-zero subrepresentations of π admit non-zero subrepresentations which are equivalent.)

More precisely, a von Neumann algebra M is called a factor if it has trivial centre – i.e., $Z(M) = M \cap M' = \mathbb{C} \cdot 1$; and von Neumann proved – see [vN3] – that any von Neumann algebra is 'a direct integral of factors'.

Projections play a major role in the theory. It is true that any von Neumann algebra is the norm-closed linear span of its projections – i.e., elements p satisfying $p = p^* = p^2$. (Reason: if $M = L^\infty(X, \mu)$, this is because simple functions – i.e., finite linear combinations of characteristic functions of sets – are dense in $L^\infty(X, \mu)$. The preceding statement and the spectral theorem show that any bounded normal operator is norm-approximable by finite linear combinations of its spectral projections. This proves the assertion about general von Neumann algebras.)

Two projections e and f in a von Neumann algebra M are said to be Murray–von Neumann equivalent – written $e \sim f$ (or $e \sim f$ (rel M)) – if there exists (a partial isometry) u in M such that $u^*u = e$ and $uu^* = f$. It is not too hard to show that if (and only if) M is a factor, any two projections in M are comparable in the sense that one is Murray–von Neumann equivalent to a sub-projection of the other.

Factors were initially classified into three broad types by Murray and von Neumann, on the basis of the structure of the lattice $\mathcal{P}(M)$ of projections in M. The key notion they use is that of a finite projection; say that a projection $e \in \mathcal{P}(M)$ is finite if e is not equivalent to any proper sub-projection of e. It should be obvious that a minimal projection of M, should one exist, is necessarily finite. It should also be equally clear that any factor is one and exactly one of the types I–III as defined below.

DEFINITION 1.1.1 *(a) A factor M is said to be of type:*

(i) I, if there exists a non-zero minimal projection in M;

(ii) II, if M contains non-zero finite projections and if M is not of type I; and

(iii) III, if no non-zero projection in M is finite.

(b) A factor M is said to be finite if 1 (the multiplicative identity of M, which always exists – cf. the definition of a concrete von Neumann algebra) is a finite projection in M; equivalently M is finite if M does not contain any non-unitary isometry.

One of the basics facts about finite factors is contained in the following result; for a proof, see, for instance [Tak1], Theorem V.2.6.

PROPOSITION 1.1.2 *If M is a finite factor, then there always exists a unique faithful normal tracial state (henceforth abbreviated simply to 'trace') on M;*

i.e., there exists a unique linear functional τ *on* M *such that:*
(i) (trace) $\tau(xy) = \tau(yx)$*;*
(ii) (state) $\tau(x^*x) \geq 0$ *and* $\tau(1) = 1$*;*
(iii) (faithful) $\tau(x^*x) \neq 0$ *if* $x \neq 0$*; and*
(iv) (normal) τ *is (*σ*-)weakly continuous.*
Further, if $e, f \in \mathcal{P}(M)$*, then*

$$e \sim f \Leftrightarrow \tau(e) = \tau(f). \tag{1.1.1}$$

We conclude this section with the rudiments of this Murray–von Neumann classification of factors. Suppose then that M is a factor with separable predual. Then consider the above-mentioned three possibilities for the type of M:

(I) If M is a factor of type I, it turns out (and it is not hard to prove – cf. [Sun1], Exercise 4.3.1, for instance) that $M \cong \mathcal{L}(\mathcal{H})$ for some separable Hilbert space $\mathcal{H}$ and that M is finite precisely when $\mathcal{H}$ is finite-dimensional. In particular, any finite factor of type I is necessarily finite-dimensional. We say that the type I factor M is of type I_n if $\mathcal{H}$ is an n-dimensional Hilbert space, for $n = 1, 2, \cdots, \infty$. If M is of type $I_n, 1 \leq n < \infty$, then $M \simeq M_n(\mathbb{C})$ and the unique trace tr of Proposition 1.1.2 is just the usual matrix-trace after suitable normalisation: thus, $\mathrm{tr}((x_{ij})) = \frac{1}{n}\sum_{i=1}^{n} x_{ii}$.

(II) If M is a factor of type II, there are two possibilities, according to whether or not M is a finite factor in the sense of Definition 1.1.1(b). We say that a type II factor M is of type II_1 or of type II_∞ according to whether or not M is a finite factor.

Recall, from Proposition 1.1.2 that every II_1 factor is equipped with a faithful normal tracial state τ. It is true – for instance, see [Sun1], Proposition 1.3.14 – that if M is a II_1 factor, then $\{\tau(p) : p \in \mathcal{P}(M)\} = [0, 1]$. Thus the trace τ induces a bijection between the collection of Murray–von Neumann equivalence classes of projections in a II_1 factor and the continuum [0,1]. What attracted von Neumann to II_1 factors was the possibility of 'continuously varying dimensions'.

On the other hand, suppose M is a II_∞ factor. Fix an arbitrary finite projection $p_1 \in M$. It is true, then, that there exists – see, for instance, [Sun1] – a sequence $\{p_n : 1 \leq n < \infty\}$ of mutually orthogonal projections in M such that (i) $p_n \sim p_1 (\mathrm{rel}\ M) \forall n$, and (ii) $\sum_n p_n = 1$. Pick a partial isometry $u_n \in M$ such that $u_n^* u_n = p_n, u_n u_n^* = p_1$. (We briefly digress to remark that it is true – see [Tak1], for instance – that if M is a von Neumann algebra, and if $p \in \mathcal{P}(M)$, then the so-called 'corner' of M defined by $M_p = pMp = \{pxp : x \in M\}$ is again a von Neumann algebra.) It then follows easily that M_{p_1} is a II_1 factor, and that the mapping $x \mapsto ((u_i^* x u_j))$ establishes an isomorphism of M onto $M_{p_1} \otimes \mathcal{L}(\ell^2)$. Now consider the map $\mathrm{Tr} : M_+ (= \{x \in M : x \geq 0\}) \to [0, \infty]$ defined by $\mathrm{Tr}\, x = \sum_n \mathrm{tr}_M(u_n x u_n^*)$. It is true of this map – see [vN1] – that

(i) $\{\mathrm{Tr}\, p : p \in \mathcal{P}(M)\} = [0, \infty]$;

(ii) if $p, q \in \mathcal{P}(M)$, then $p \sim q(\text{rel } M) \Leftrightarrow \operatorname{Tr} p = \operatorname{Tr} q$; and

(iii) if $p, q \in \mathcal{P}(M)$, and $pq = 0$, then $\operatorname{Tr}(p+q) = \operatorname{Tr} p + \operatorname{Tr} q$.

Notice that this choice of Tr is so 'normalised' that it takes the value 1 at p_1; it is true that but for this possible choice in scaling, the function Tr is uniquely determined by the above properties. This function Tr is referred to as the faithful normal semifinite trace on the II_∞ factor.

(III) The factor M is of type III precisely when every non-zero projection is infinite; under our standing assumption on the separability of the pre-dual of M, it turns out – cf. [Sun1], Corollary 1.2.4(b) – that any two infinite projections are equivalent. Thus M is a type III factor precisely when any non-zero projection is Murray–von Neumann equivalent to the identity 1.

1.2 The standard form

Suppose φ is a *normal state* on a von Neumann algebra M – i.e., φ is a linear functional on M which is:

(i) positive – i.e., $\varphi(x^*x) \geq 0$;

(ii) a state – $\varphi(1) = 1$;

(iii) normal – i.e., φ is σ-weakly continuous.

Consider the sesquilinear form on M defined by $(x, y) \to \langle x, y \rangle = \varphi(y^*x)$. This satisfies all the requirements of an inner product except positive definiteness – i.e., the set $N_\varphi = \{x \in M : \varphi(x^*x) = 0\}$ may be non-trivial. It is, in any case, a consequence of the Cauchy–Schwarz inequality that N_φ is a left-ideal of M. Hence the form $\langle \cdot, \cdot \rangle$ descends to a genuine inner product on the quotient space M/N_φ; further, the equation $\pi(x)(y + N_\varphi) = xy + N_\varphi$ defines, not just a well-defined, but even a bounded – with respect to the norm $||y + N_\varphi||_2 = \varphi(y^*y)^{1/2}$ – linear operator $\pi(x)$ on M/N_φ, which hence extends to the completion $\mathcal{H}_\varphi$. It is painless to verify that if $\pi_\varphi(x)$ denotes the extension to $\mathcal{H}_\varphi$ of $\pi(x)$, then π_φ defines a normal representation of M on $\mathcal{H}_\varphi$. Further, if $\xi_\varphi = 1 + N_\varphi$, then ξ_φ is a cyclic vector for π_φ, i.e., $[\pi_\varphi(M)\xi_\varphi] = \mathcal{H}_\varphi$ (where $[S]$ denotes the closed subspace spanned by a subset S of Hilbert space), and the given state φ is recovered from the triple $(\mathcal{H}_\varphi, \pi_\varphi, \xi_\varphi)$ by

$$\varphi(x) = \langle \pi_\varphi(x)\xi_\varphi, \xi_\varphi \rangle, x \in M.$$

We summarise the foregoing construction – called the GNS construction after Gelfand, Naimark and Segal – in the following:

PROPOSITION 1.2.1 *Let φ be a normal state on a von Neumann algebra M. Then there exists a triple $(\mathcal{H}, \pi, \xi)$ consisting of a Hilbert space $\mathcal{H}$ carrying a normal representation π of M, and a distinguished cyclic vector ξ of the representation satisfying $\varphi(x) = \langle \pi(x)\xi, \xi \rangle$ for all x in M.* □

Notice, incidentally, that if the state φ is faithful (meaning $\varphi(x^*x) \neq 0$ if $x \neq 0$), so is the representation π.

EXAMPLE 1.2.2 *If $M = L^\infty(X, \mu)$, a typical normal state on M is of the form $\phi_\nu(f) = \int f d\nu, f \in L^\infty(X, \mu)$ – where ν is a probability measure on X which is absolutely continuous with respect to μ. One GNS triple associated to ϕ_ν is given by $\mathcal{H}_\nu = L^2(X, \nu), \pi_\nu(f)\xi = f\xi$, and ξ_ν is the constant function* 1. □

It must be clear that a necessary and sufficient condition for a normal representation π of M to occur as a GNS representation (i.e., to be unitarily equivalent to one) is that the representation π be a cyclic representation.

An easy application of Zorn's lemma shows now that every (separable) normal representation of a von Neumann algebra is a (countable) direct sum of GNS representations.

In the rest of this section, we consider the special case when M admits a faithful normal tracial state, and analyse the GNS construction.

Assume thus that there exists a faithful normal tracial state τ on M. We write $L^2(M, \tau)$ for the Hilbert space underlying the GNS representation associated with τ. This representation is faithful, since $\pi(x) = 0 \Rightarrow 0 = ||\pi(x)\xi_\tau||^2 = \tau(x^*x) \Rightarrow x = 0$. Hence we may – and do – identify x with $\pi_\tau(x)$ and we assume that $M \subseteq \mathcal{L}(L^2(M, \tau))$ so that there exists a cyclic vector Ω (which is the notation we shall employ for what we earlier called ξ_τ) such that $\tau(x) = \langle x\Omega, \Omega \rangle$ for all $x \in M$.

Besides being a cyclic vector for M, the vector Ω is also separating for M in the sense that $x\Omega = 0 (\Rightarrow \tau(x^*x) = ||x\Omega||^2 = 0) \Rightarrow x = 0$.

Hence the Hilbert space $L^2(M, \tau)$ contains a vector Ω which is simultaneously cyclic and separating for M. We need the fact – which is ensured by the next lemma – that Ω is also a cyclic and separating vector for M'.

LEMMA 1.2.3 *If $M \subseteq \mathcal{L}(\mathcal{H})$ is a von Neumann algebra, a vector ξ in $\mathcal{H}$ is cyclic for M if and only if ξ is separating for M'.*

Proof : Let $\xi \in \mathcal{H}$. Let p' be the projection onto $[M\xi]$, the closed M-cyclic subspace spanned by ξ. Note that $p' \in M'$ and that $(1 - p')\xi = 0$. So, if ξ is separating for M', then $p' = 1$ so ξ is cyclic for M.

Conversely if ξ is cyclic for M, then $x' \in M'$ and $x'\xi = 0 \Rightarrow x'\eta = 0$ for all η in $[M\xi]$, since $x'(x\xi) = x(x'\xi) = 0$ for x in M. □

Thus $L^2(M, \tau)$ has the vector Ω which is cyclic and separating for M as well as for M'. For any vector ξ in L^2, consider the two operators defined by:

$$\pi_\ell(\xi)(x'\Omega) = x'\xi \quad \forall x' \in M', \tag{1.2.1}$$
$$\pi_r(\xi)(x\Omega) = x\xi \quad \forall x \in M. \tag{1.2.2}$$

Thus, dom $\pi_\ell(\xi) = M'\Omega$ (and dom $\pi_r(\xi) = M\Omega$); these operators are densely and unambiguously defined since Ω is a cyclic and separating vector for M and for M'.

Call a vector ξ *left-bounded* (respectively *right-bounded*) if the operator $\pi_\ell(\xi)$ (resp., $\pi_r(\xi)$) extends to a (necessarily unique) bounded operator on all of $\mathcal{H}$ (or equivalently, is bounded on its dense domain of definition).

If ξ is left- (resp. right-) bounded, we shall continue to write $\pi_\ell(\xi)$ (resp. $\pi_r(\xi)$) for the unique continuous extension to all of $L^2(M,\tau)$.

In order to state the fundamental proposition concerning $L^2(M,\tau)$ – or the standard form of the finite von Neumann algebra M, as it is referred to – we need one last bit of notation: the map $x\Omega \mapsto x^*\Omega$ is a conjugate-linear isometry from $M\Omega \subseteq L^2(M,\tau)$ onto itself. Denote its extension to $L^2(M,\tau)$ by J. It must be clear that J is an anti-unitary involution – i.e., J is a conjugate linear isometry of $L^2(M,\tau)$ onto itself whose square is the identity; in particular $J = J^* = J^{-1}$. This operator J is referred to as the *modular conjugation operator* for M, and sometimes denoted by J_M.

THEOREM 1.2.4 *(1)* $JMJ = M'$.

(2) The following conditions on a vector $\xi \in L^2(M,\tau)$ *are equivalent:*
(i) ξ *is left-bounded;*
(i)' $\xi = x\Omega$ *for some (uniquely determined) element* x *of* M*;*
(ii) ξ *is right-bounded;*
(ii)' $\xi = x'\Omega$ *for some (uniquely determined) element* x' *of* M'.

Proof: (1) The definition of J shows that if $x, y \in M$, then $(Jx^*J)(y\Omega) = yx\Omega$; since 'left multiplications commute with right multiplications', we find that

$$JMJ \subseteq M'. \tag{1.2.3}$$

On the other hand, if $x \in M, x' \in M'$, then by the 'self-adjointness' of the anti-unitary operator J, we have

$$\begin{aligned}
\langle Jx'\Omega, x\Omega\rangle &= \langle Jx\Omega, x'\Omega\rangle \\
&= \langle x^*\Omega, x'\Omega\rangle \\
&= \langle \Omega, xx'\Omega\rangle \\
&= \langle \Omega, x'x\Omega\rangle \\
&= \langle x'^*\Omega, x\Omega\rangle.
\end{aligned}$$

Since x was arbitrary, this implies that

$$Jx'\Omega = x'^*\Omega, \ \forall x' \in M'. \tag{1.2.4}$$

The above equation, together with the same reasoning that led to (1.2.3), now shows that

$$JM'J \subseteq M, \tag{1.2.5}$$

and the equality in (1) follows from equations (1.2.3) and (1.2.5).

(2) Denote the set of left- (resp., right-) bounded vectors in $L^2(M,\tau)$ by $\mathcal{U}_\ell$ (resp., $\mathcal{U}_r$); it is clear from the definitions that $\xi \in \mathcal{U}_\ell \Rightarrow \pi_\ell(\xi) \in (M')' = M$ and $\xi = \pi_\ell(\xi)\Omega$; conversely, if $x \in M$, then $x\Omega \in \mathcal{U}_\ell$ and $x = \pi_l(x\Omega)$. Thus the map $\xi \mapsto \pi_\ell(\xi)$ establishes a bijective linear map of $\mathcal{U}_\ell$ onto M, with inverse being given by $x \to x\Omega$. In an identical manner, we also have $\mathcal{U}_r = M'\Omega$. Thus $(i) \Leftrightarrow (i)'$ and $(ii) \Leftrightarrow (ii)'$.

To complete the proof, we only need to prove that $M\Omega = M'\Omega$. If $x \in M$, then

$$x\Omega = Jx^*J\Omega \in M'\Omega,$$

hence showing that $M\Omega \subseteq M'\Omega$. An identical reasoning, with the roles of M and M' reversed, proves the reverse inclusion, and hence the theorem. □

1.3 Discrete crossed products

The two constructions used initially by Murray and von Neumann – see [vN1] and [vN2] – to construct examples of factors of all possible types were (i) the crossed product construction, and (ii) the infinite tensor product construction. This section is devoted to a discussion of the former, while the latter will be discussed at the end of this chapter.

The starting data is a (discrete) group G acting on a von Neumann algebra M – i.e., suppose we are given a group homomorphism $t \mapsto \alpha_t$ from G to the group Aut(M) of (normal) *-automorphisms of M. (It is a fact (cf. [Sun1], Exercise 2.3.4(a)) that all *-algebra automorphisms of a von Neumann algebra are automatically normal.) The crossed product is a sort of maximal von Neumann algebra containing copies of M and G with 'commutation relations governed by the given action of G on M'; a little more precisely, the crossed product of M by the action α of G is a specific von Neumann algebra of the form $\tilde{M} = (\pi(M) \cup \lambda(G))''$ where $\pi : M \to \tilde{M}$ (resp., $\lambda : G \to \mathcal{U}(\tilde{M})$, the unitary group of $\tilde{M}$) is an injective normal *-homomorphism (resp., injective unitary representation) of M (resp., of G), such that the copies $\pi(M)$ and $\lambda(G)$ of M and G satisfy the commutation relations

$$\lambda(t)\pi(x)\lambda(t)^* = \pi(\alpha_t(x)), \quad \text{for all } \; x \in M, t \in G. \tag{1.3.1}$$

The construction of the crossed product (which is usually denoted by $M \times_\alpha G$, or simply $M \times G$) goes as follows:

Suppose $M \subseteq \mathcal{L}(\mathcal{H})$. The Hilbert space $\tilde{\mathcal{H}}$ on which $\tilde{M}$ will be represented has three (unitarily equivalent) descriptions:

(i) $\tilde{\mathcal{H}} = \ell^2(G;\mathcal{H}) = \{\xi : G \to \mathcal{H} | \sum_{t\in G} ||\xi(t)||^2 < \infty\}$;

(ii) $\tilde{\mathcal{H}} = \bigoplus_{t\in G} \mathcal{H} = \{((\xi(t)))_{t\in G} : \sum_{t\in G} ||\xi(t)||^2 < \infty\}$

where we think of a typical element of $\tilde{\mathcal{H}}$ as a column-vector with norm-square-summable entries from $\mathcal{H}$; although this seems an artificial variant of (i), it is this description, in view of the availability of the convenience of matrix manipulations, which will prove most useful for dealing with $M \times_\alpha G$;

(iii) $\tilde{\mathcal{H}} = \mathcal{H} \otimes \ell^2(G)$, the Hilbert space tensor product of $\mathcal{H}$ and the Hilbert space $\ell^2(G)$ of square-summable complex functions on G (or $\ell^2(G;\mathbb{C})$ in the notation of (i)).

Before proceeding further, let us remark that $\ell^2(G)$ carries two natural – the so-called left-regular and right-regular – unitary representations $\lambda, \rho : G \to \mathcal{U}(\mathcal{L}(\ell^2(G)))$, defined by $(\lambda_u \xi)(t) = \xi(u^{-1}t)$ and $(\rho_u \xi)(t) = \xi(tu)$. Alternatively, if $\{\xi_t : t \in G\}$ denotes the canonical (or distinguished) orthonormal basis of $\ell^2(G)$ – thus ξ_t is the characteristic function of the singleton set $\{t\}$ – then $\lambda_u(\xi_t) = \xi_{ut}$ and $\rho_u(\xi_t) = \xi_{tu^{-1}}$. It is a basic fact – see §1.4 – that

$$\lambda(G)' = (\rho(G))'', \rho(G)' = \lambda(G)''. \tag{1.3.2}$$

To return to the definition of the crossed product, define $\pi : M \to \mathcal{L}(\tilde{\mathcal{H}})$ and $\lambda : G \to \mathcal{L}(\tilde{\mathcal{H}})$ by:

$$\left.\begin{array}{rcl}(\pi(x)\tilde{\xi})(t) & = & \alpha_{t^{-1}}(x)\tilde{\xi}(t), \\ (\lambda(u)\tilde{\xi})(t) & = & \tilde{\xi}(u^{-1}t).\end{array}\right\} \tag{1.3.3}$$

It is trivial to verify that $\pi : M \to \mathcal{L}(\tilde{\mathcal{H}})$ (resp. $\lambda : G \to \mathcal{L}(\tilde{\mathcal{H}})$) is a faithful normal *-homomorphism (resp., faithful unitary representation) and that π and λ satisfy equation (1.2.1). Now define $\tilde{M} = (\pi(M) \cup \lambda(G))''$, so that $\tilde{M}$ is the smallest von Neumann subalgebra of $\mathcal{L}(\tilde{\mathcal{H}})$ containing $\pi(M)$ and $\lambda(G)$. This $\tilde{M}$ is, by definition, the crossed product $M \times_\alpha G$.

We used some realisation of M as a concrete von Neumann algebra to define the crossed product $M \times_\alpha G$. It is true, however – cf., for instance, [Sun1], Proposition 4.4.4 – that the isomorphism class of the von Neumann algebra $M \times_\alpha G$ so constructed does not depend upon which faithful realisation M on Hilbert space one started with.

We shall now pass to a closer analysis of $\tilde{M}$, by using the second picture of $\tilde{\mathcal{H}}$ as $\bigoplus_{t \in G} \mathcal{H}$. In this form, it is clear that any bounded operator $\tilde{x} \in \mathcal{L}(\tilde{\mathcal{H}})$ is represented by a matrix $\tilde{x} = ((\tilde{x}(s,t)))_{s,t \in G}$ where $\tilde{x}(s,t) \in \mathcal{L}(\mathcal{H})$ for all $s, t \in G$, and $(\tilde{x}\tilde{\xi})(s) = \sum_{t \in G} \tilde{x}(s,t)\tilde{\xi}(t)$, the sum on the right being intepreted as the norm limit of the net of finite sums. In this language, it is clear that

$$(\pi(x))(s,t) = \delta_{st}\alpha_{t^{-1}}(x)$$

and

$$(\lambda(u))(s,t) = \delta_{s,ut}$$

for x in M, u, s, t in G.

The point is that while elements of $\mathrm{Mat}_G(\mathbb{C})$ have two degrees of freedom, the above generators of the crossed product have only one degree of freedom. More precisely, we have the following matricial description of the crossed product (where we identify an element $\tilde{x} \in \mathcal{L}(\tilde{\mathcal{H}})$ with its matrix $((\tilde{x}(s,t)))$ (with respect to the orthonormal basis $\{\xi_t : t \in G\}$)).

Lemma 1.3.1 *With the preceding notation, we have* $\tilde{M} = \{\tilde{x} \in \mathcal{L}(\tilde{\mathcal{H}}) : \exists x : G \to M \ \text{ s.t. } \ \tilde{x}(s,t) = \alpha_{t^{-1}}(x(st^{-1})) \ \forall s,t \in G\}$.

Proof: On the one hand, the right side is seen, quite easily, to define a (weakly closed self-adjoint algebra of operators and hence a) von Neumann subalgebra of $\mathcal{L}(\tilde{\mathcal{H}})$, while on the other, the algebra generated by $\pi(M) \cup \lambda(G)$ is seen to be the dense subalgebra, corresponding to finitely supported functions, of the one given by the right side. □

It follows from Lemma 1.3.1 that the crossed product $M \times_\alpha G$ is identifiable – via the association $\tilde{x} \mapsto x(s) = \tilde{x}(s,1)$ – with a space of functions from G to M – viz.

$$\tilde{M} = \{x : G \to M | \exists \tilde{x} \in \mathcal{L}(\tilde{\mathcal{H}}) \text{ s.t. } \tilde{x}(s,t) = \alpha_{t^{-1}}(x(st^{-1})) \ \forall s,t \in G\}. \quad (1.3.4)$$

It is a matter of easy verification to check that the algebra structure inherited from $(\pi(M) \cup \lambda(G))''$ by the set $\tilde{M}$ defined by equation (1.3.4) is:

$$\left.\begin{aligned} (x * y)(s) &= \sum_{t \in G} \alpha_{t^{-1}}(x(st^{-1}))y(t), \\ x^*(s) &= \alpha_{s^{-1}}(x(s^{-1})^*). \end{aligned}\right\} \quad (1.3.5)$$

(The series above is interpreted as the limit, in the weak topology, of the net of finite sums; this converges by the nature of matrix multiplication.)

In the new notation, note that

$$\pi(x)(s) = \delta_{s1} \cdot x, x \in M,$$

and

$$\lambda(u)(s) = \delta_{su} \cdot 1, u \in G.$$

We conclude this section by determining when the crossed product is a finite von Neumann algebra – i.e., admits a faithful normal tracial state.

Proposition 1.3.2 *Let $\tilde{M} = M \times_\alpha G$ be as above (cf. the discussion preceding equation (1.3.5)). Then $\tilde{M}$ admits a faithful normal tracial state $\tilde{\tau}$ if and only if M admits a faithful normal tracial G-invariant state τ (where G-invariance means $\tau \circ \alpha_t = \tau \ \ \forall t \in G$).*

Proof: If $\tilde{\tau}$ is a faithful normal tracial state on $\tilde{M}$, then $\tau(x) = \tilde{\tau}(\pi(x))$ defines a faithful normal tracial state on M which is G-invariant since $\tau(\alpha_t(x)) = \tilde{\tau}(\pi(\alpha_t(x))) = \tilde{\tau}(\lambda(t)\pi(x)\lambda(t)^{-1}) = \tilde{\tau}(\pi(x)) = \tau(x)$.

Conversely, if τ is a G-invariant faithful normal tracial state, define $\tilde{\tau}(x) = \tau(x(1))$ $(= \tau(\tilde{x}(1,1)))$. Then $\tilde{\tau}$ is clearly a normal positive linear function; further, $\tilde{\tau}$ is faithful (since $\tilde{x} \in \tilde{M}_+$ and $\tau(\tilde{x}(1,1)) = 0$ implies $\tilde{x}(1,1) = 0$, whence $\tilde{x}(t,t) = \alpha_{t^{-1}}(\tilde{x}(1,1)) = 0$ for all t in G, whence $\tilde{x} = 0$ (since a positive operator with zero diagonal is zero)). Finally, $\tilde{\tau}$ is a trace, since

$$\begin{aligned}
\tilde{\tau}(x * y) &= \tau((x * y)(1)) \\
&= \tau(\sum_{t\in G} \alpha_{t^{-1}}(x(t^{-1}))y(t)) \\
&= \sum_{t\in G} \tau(\alpha_{t^{-1}}(x(t^{-1}))y(t)) \\
&\quad \text{(since } \tau \text{ is normal)} \\
&= \sum_{t\in G} \tau(x(t^{-1})\alpha_t(y(t))) \\
&\quad \text{(since } \tau \text{ is } G\text{-invariant)} \\
&= \sum_{t\in G} \tau(\alpha_t(y(t))x(t^{-1})) \\
&\quad \text{(since } \tau \text{ is a trace)} \\
&= \sum_{s\in G} \tau(\alpha_{s^{-1}}(y(s^{-1}))x(s)) \\
&= \tilde{\tau}(y * x).
\end{aligned}$$

□

Before discussing when the crossed product is a factor, we digress for some examples, one of which will motivate the definitions of the necessary concepts.

1.4 Examples of factors

1.4.1 Group von Neumann algebras

In the notation of §1.2, the (left) group von Neumann algebra LG of the discrete group G is defined thus:

$$LG = \mathbb{C} \times G = \lambda(G)'' \subseteq \mathcal{L}(\ell^2(G)),$$

where the crossed product is with respect to the trivial action – $\alpha_t(z) = z, \ \forall t \in G, z \in \mathbb{C}$ – of G on $\mathbb{C}$, and, of course, λ denotes the left-regular representation of G on $\ell^2(G)$.

The analysis of §1.2 translates, in this most trivial case of a crossed product, as follows: the elements of LG are those $\tilde{x} \in \mathcal{L}(\ell^2(G))$ whose matrix, with respect to the standard orthonormal basis $\{\xi_t : t \in G\}$ of $\ell^2(G)$, has