# 7

# Some Auxiliary Results on Abstract Equations

In the present chapter we collect, in Sections 7.1 through 7.5, a number of results that will be invoked repeatedly throughout the present volume as well as Volume III. They concern (1) regularity results of the input → solution map, and its adjoint map, over both a finite or an infinite time interval and (2) generation and abstract trace regularity under unbounded perturbation. In addition, in Section 7.6, we provide an abstract regularity result for damped second-order equations of interest in itself. Illustrations thereof are given in Section 7.7 and in Chapter 9, Section 9.10.4.

## 7.1 Mathematical Setting and Standing Assumptions

Throughout this chapter $X$ and $U$ are reflexive Banach spaces and $X^*$ and $U^*$ are their dual spaces. For a given $0 < T < \infty$ fixed, we shall study the operator $L$,

$$(Lu)(t) = \int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau, \quad 0 \le t \le T, \tag{7.1.1}$$

corresponding to the mild solution

$$x(t) = e^{At}x_0 + (Lu)(t) \tag{7.1.2}$$

of the abstract equation

$$\dot{x} = Ax + Bu \in [\mathcal{D}(A^*)]', \quad x(0) = x_0, \tag{7.1.3}$$

subject to the following standing assumptions:

(H.1) $A : X \supset \mathcal{D}(A) \to X$ is a linear operator, which is the infinitesimal generator of a strongly continuous (s.c.) semigroup $e^{At}$ on $X$.

(H.2) $B$ is a linear, continuous operator $U \to [\mathcal{D}(A^*)]'$, where $A^*$ is the $X$-adjoint of $A$, and $[\mathcal{D}(A^*)]'$ is the dual of $\mathcal{D}(A^*)$ with respect to the pivot space $X$, or equivalently,

$$A^{-1}B \in \mathcal{L}(U; X). \tag{7.1.4a}$$

For considerations of $L$, over a finite time interval, we may assume, without loss of generality, that $A^{-1} \in \mathcal{L}(X)$, for otherwise we replace (7.1.4a) with

$$(\lambda_0 I - A)^{-1} B \in \mathcal{L}(U; X), \quad \lambda_0 \in \text{resolvent set } \rho(A). \tag{7.1.4b}$$

(H.3) (Abstract trace regularity) Given $0 < T < \infty$ and $1 \leq q \leq \infty$, there exists a constant $C_T > 0$ [which depends on $T$ and on $q$, but dependence on $q$ is omitted], such that

$$\begin{cases} \int_0^T \|B^* e^{A^* t} x^*\|_{U^*}^q \, dt \leq C_T \|x^*\|_{X^*}^q, & x^* \in \mathcal{D}(A^*), \ 1 \leq q < \infty; \\ \|B^* e^{A^* t} x^*\|_{L_\infty(0,T;U^*)} \leq C_T \|x^*\|_{X^*}, & x^* \in \mathcal{D}(A^*), \ q = \infty, \end{cases} \tag{7.1.5a}$$

so that the closable (cf. Remark 7.1.1 below) operator $B^* e^{A^* t}$ admits a continuous extension (which may then be denoted by the same symbol) satisfying

$$B^* e^{A^* t} : \text{ continuous } X^* \to L_q(0, T; U^*), \quad 1 \leq q \leq \infty, \tag{7.1.5b}$$

that is,

$$\begin{cases} \int_0^T \|B^* e^{A^* t} x^*\|_{U^*}^q \, dt \leq C_T \|x^*\|_{X^*}^q, & x^* \in X^*, \ 1 \leq q < \infty; \\ \|B^* e^{A^* t} x^*\|_{L_\infty(0,T;U)} \leq C_T \|x^*\|_{X^*}, & x^* \in X^*, \ q = \infty. \end{cases} \tag{7.1.5c}$$

Here $B^*$, the dual of $B$, satisfies $B^* \in \mathcal{L}(\mathcal{D}(A^*); U^*)$, after identifying $[\mathcal{D}(A^*)]''$ with $\mathcal{D}(A^*)$. Moreover, $e^{A^* t}$ is a s.c. semigroup on $X^*$.

***Remark 7.1.1*** By a change of variable and use of the semigroup property, if (7.1.5) holds for one fixed $0 < T < \infty$, then (7.1.5) holds for $2T, 3T, \ldots$, hence for *any* $0 < T < \infty$.

***Remark 7.1.2*** The original s.c. semigroup $e^{At}$ on $X$ of assumption (H.1) can always be extended as a s.c. semigroup on the extrapolation space $[\mathcal{D}(A^*)]'$. We shall continue to use the notation $e^{At}$ for such an extension. Moreover, solely under (H.1) and (H.2), it is plainly always the case that

$$L : \text{ continuous } L_p(0, T; U) \to C([0, T]; [\mathcal{D}(A^*)]'), \quad 1 \leq p \leq \infty. \tag{7.1.6}$$

***Remark 7.1.3*** Under (H.1) and (H.2), if $u \in H^1(0, T; U)$, then formula (7.1.1) for $L$ yields, after integration by parts,

$$\begin{aligned} (Lu)(t) &= -\int_0^t \frac{de^{A(t-\tau)}}{d\tau} A^{-1} B u(\tau) \, d\tau \\ &= e^{At} A^{-1} B u(0) - A^{-1} B u(t) + \int_0^t e^{A(t-\tau)} A^{-1} B \dot{u}(\tau) \, d\tau \\ &\in C([0, T]; X). \end{aligned} \tag{7.1.7}$$

The above computations are justified on the extrapolation space $[\mathcal{D}(A^*)]'$, via Remark 7.1.2. However, the final result lies in $X$, at least for $u \in H^1(0, T; U)$, as noted in (7.1.7). Thus, (7.1.7) says, in particular, that the operator

$$L_T u = \int_0^T e^{A(T-t)} Bu(t)\, dt = A \int_0^T e^{A(T-t)} A^{-1} Bu(t)\, dt \tag{7.1.8}$$

is densely defined as an operator $L_p(0, T; U) \supset \mathcal{D}(L_T) \to X$, $1 \le p \le \infty$. Moreover, $L_T$ – being in (7.1.8) (right) the composition of a closed boundedly invertible operator $A$ with a bounded operator – is closed [Kato, 1966, p. 164]. An adjoint of $L_T$ is:

$$(Sx^*)(t) = B^* e^{A^* t} x^*, \quad x^* \in \mathcal{D}(S), \ 0 \le t \le T, \text{ a.e.}, \tag{7.1.9a}$$

which is closable [Kato, 1966, p. 168] as an operator $X^* \supset \mathcal{D}(S) \to L_q(0, T; U^*)$, $1 \le q \le \infty$, as noted above (7.1.5b), where $1/p + 1/q = 1$. The unique maximal extension of the operator in (7.1.9a), which is mentioned in assumption (H.3) above, is the adjoint $L_T^*$ of $L_T$. Since $L_T$ in (7.1.8) is densely defined and closed, then $(L_T^*)^* = L_T$. Thus, hypothesis (H.3) = (7.1.5b) is now paraphrased by saying

$$(L_T^* x^*)(t) = B^* e^{A^* t} x^* : \text{ continuous } X^* \to L_q(0, T; U^*), \tag{7.1.9b}$$

or equivalently for $1 \le q < \infty$, and a fortiori for $q = \infty$:

(H.3*)

$$L_T : \text{continuous } L_p(0, T; U) \to X, \quad 1 \le p \le \infty, \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{7.1.10}$$

The point is that, although the extrapolation space $[\mathcal{D}(A^*)]'$ acts as an all-encompassing, backup space for the regularity of $L$, and for performing computations, solely under (H.1) and (H.2), according to (7.1.7), $L$ may be more regular. Indeed, more precisely, we have that

$$L : \text{continuous } L_p(0, T; U) \to C([0, T]; X), \tag{7.1.11}$$

with $1 \le p < \infty$, if and only if (H.3) holds true with $1 < q \le \infty$, or, with $p = \infty$, provided that (H.3) holds true with $q = 1$. This is the content of Theorem 7.2.1 below.

***Remark 7.1.4*** We note explicitly that, if there exists a point $T$, $0 < T < \infty$, such that

$$L_T u = \int_0^T e^{A(T-\tau)} Bu(\tau)\, d\tau \in X, \quad \forall\, u \in L_2(0, T; U), \tag{7.1.12}$$

then, for all $0 < t_1 < T$, we likewise have

$$L_{t_1} u = \int_0^{t_1} e^{A(t_1-\tau)} Bu(\tau)\, d\tau \in X, \quad \forall\, u \in L_2(0, t_1; U). \tag{7.1.13}$$

Indeed, choosing at first $u$ smooth, say $u \in C([0,T];U)$, we write for any $0<t<T$:

$$L_T u = \int_0^t e^{A(T-\tau)} Bu(\tau)\,d\tau + \int_t^T e^{A(T-\tau)} Bu(\tau)\,d\tau \tag{7.1.14}$$

$$= \int_0^T e^{A(T-\tau)} Bu_{\text{ext}}(\tau)\,d\tau + \int_0^{T-t} e^{A(T-t-\sigma)} Bu_t(\sigma)\,d\sigma, \tag{7.1.15}$$

where $u_{\text{ext}}(\tau)$ extends by zero $u(\tau)$ for $t < \tau \le T$ in the first integral, whereas in the second integral we set $u_t = u(t+\sigma)$, $\tau - t = \sigma$. Extending $u$ to all $u \in L_2(0,T;U)$ and setting $t_1 = T - t$, we obtain, by assumption (1.12),

$$L_{t_1} u_t = L_T u - L_T u_{\text{ext}} \in Y, \quad \forall\, u \in L_2(0, T_1; U), \tag{7.1.16}$$

as desired, and (7.1.13) is established.

## 7.2 Regularity of $L$ and $L^*$ on $[0, T]$

As anticipated at the end of Remark 7.1.3, the "trace" regularity (H.3) = (7.1.5) of $L_T^*$, equivalently the final state "interior" regularity (H.3*) = (7.1.10) of $L_T$, is equivalent to the following "interior" regularity for $L$.

***Theorem 7.2.1*** *Assume (H.1) and (H.2).*

*(i) For $1<q\le\infty$ [respectively, $q=1$], hypothesis (H.3) is equivalent to [respectively, implies] the following regularity property of the operator $L$ in (7.1.1):*

$$L: \text{ continuous } L_p(0,T;U) \to C([0,T];X), \quad 1 \le p \le \infty, \tag{7.2.1a}$$

*that is, there exists $k_T > 0$ such that*

$$\|Lu\|_{C([0,T];X)} \le k_T \|u\|_{L_p(0,T;U)}. \tag{7.2.1b}$$

*(ii) For $1 \le q \le \infty$, hypothesis (H.3) implies that the adjoint operator $L^*$ satisfies*

$$(L^* v)(t) = \int_t^T B^* e^{A^*(\tau - t)} v(\tau)\,d\tau \tag{7.2.2a}$$

$$: \text{continuous } L_1(0,T;X^*) \to L_q(0,T;U^*). \tag{7.2.2b}$$

*The operator $L^*$ is the adjoint of $L$ in the sense that, with $1/p + 1/q = 1$,*

$$(Lu, v)_{p,X;q,X^*} = (u, L^* v)_{p,U;q,U^*}, \tag{7.2.3}$$

*where the notation on the left and on the right of (7.2.3) denotes, respectively, the duality pairing between $L_p(0,T;X)$ and $L_q(0,T;X^*)$, and between $L_p(0,T;U)$ and $L_q(0,T;U^*)$.*

*Proof.*

***(i) Step 1*** We first prove that, under (H.1) and (H.2), assumption (H.3) = (7.1.5) for $1 \leq q \leq \infty$ implies that

$$L: \text{ continuous } L_p(0, T; U) \to L_\infty(0, T; X). \tag{7.2.4}$$

To this end, with $v \in L_1(0, T; X^*)$ and $u \in L_p(0, T; U)$, we compute from (7.2.3) and (7.1.1), with $1 < p < \infty$: To this end, with $v \in L_1(0, T; X^*)$ and $u \in L_p(0, T; U)$, we compute from (7.2.3) and (7.1.1), with $1 < p < \infty$:

$$|(Lu, v)_{\infty, X; 1, X^*}| = \left| \int_0^T ((Lu)(t), v(t))\, dt \right|$$

$$= \left| \int_0^T \int_0^t \left(u(\tau), B^* e^{A^*(t-\tau)} v(t)\right) d\tau\, dt \right|$$

$$(t - \tau = \sigma) \qquad \leq \int_0^T \left\{ \int_0^t \|u(\tau)\|_U^p\, d\tau \right\}^{\frac{1}{p}} \left\{ \int_0^t \|B^* e^{A^*\sigma} v(t)\|_{U^*}^q\, d\sigma \right\}^{\frac{1}{q}} dt$$

(replacing $t$ with $T$ in both integral signs, and using (H.3) = (7.1.5c))

$$\leq \|u\|_{L_p(0,T;U)} \int_0^T C_T \|v(t)\|_{X^*}\, dt$$

$$= C_T \|u\|_{L_p(0,T;U)} \|v\|_{L_1(0,T;X^*)}. \tag{7.2.5}$$

[In the above estimates, either one considers $v(t) \in X^*$ a.e. or one takes $v(t) \in C([0, T]; X^*)$ and extends estimate (7.2.5) to all of $v \in L_1(0, T; X^*)$.] An obvious variation leads likewise to (7.2.5) also for $p = \infty$, or $q = \infty$. Then (7.2.4) is established for $1 \leq q \leq \infty$.

***Step 2*** Let $u \in L_p(0, T; U)$. By taking now $u_n$ smooth, say $u_n \in C^1([0, T]; U)$, with $u_n \to u$ in $L_p(0, T; U)$, and integrating $(Lu_n)(t)$ by parts as in (7.1.7), we then see that $(Lu_n)(t) \in C([0, T]; X)$ by (7.1.7), whereas $Lu_n \to Lu$ in $L_\infty(0, T; X)$ by (7.2.4), and thus $Lu \in C([0, T]; X)$. The continuity of $L$ in (7.2.4) is then improved to the continuity of $L$ in (7.2.1), as desired.

***Step 3*** Conversely, the continuity of $L$ in (7.2.1) implies the continuity of $L_T$ in (7.1.10), and hence, equivalently for $1 \leq p < \infty$, the continuity of $L_T^*$ defined by (7.1.9) given by (H.3) = (7.1.5b).
***(ii)*** Statement (7.2.2b) follows from (7.2.1) by duality. □

***Remark 7.2.1*** In application to partial differential equations in this volume we shall use Theorem 7.2.1 in the Hilbert space setting, with $X = X^*$ and $U = U^*$ Hilbert spaces, and $p = q = 2$.

***Remark 7.2.2*** The conclusion of Theorem 7.2.1 for $L$ applies also to the operator

$$u \to \int_t^T e^{A(\tau - t)} Bu(\tau)\, d\tau, \tag{7.2.6}$$

as we shall need, for example, in Chapters 8 and 9. Similarly, at times, as in the forthcoming Chapter 8, Section 8.2.1, we shall start with the assumption

$$Re^{At}B : \text{continuous } U \to L_q(0, T; Z), \tag{7.2.7}$$

for $A$ and $B$ as in (H.1) and (H.2) = (7.1.4), and with $R$ a suitable operator $R \in \mathcal{L}(Y; Z)$ and $U$ and $Z$ Hilbert spaces [the case $q = 1$ will be relevant]. Then, a variation of the proof of Theorem 7.2.1 yields: For $1 < q \leq \infty$ [respectively, for $q = 1$], property (7.2.7) is equivalent to [respectively, implies] the following property:

$$(L^* R^* f)(t) = \int_t^T B^* e^{A^*(\tau - t)} R^* f(\tau)\, d\tau \tag{7.2.8a}$$

$$: \text{continuous } L_p(0, T; Z) \to C([0, T]; U), \tag{7.2.8b}$$

where $1/p + 1/q = 1$. Indeed, the counterpart of Step 1 in (7.2.5) is now, by (7.2.8a), for $f \in L_p(0, T; Z)$, $1 \leq p < \infty$, $g \in L_1(0, T; U)$,

$$\begin{aligned} |(L^* R^* f, g)_{\infty, U; 1, U}| &= \left| \int_0^T \int_t^T \left( f(\tau), Re^{A(\tau - t)} Bg(t) \right)_Z d\tau\, dt \right| \\ &\leq \int_0^T \left\{ \int_t^T \| f(\tau) \|_Z^p \, d\tau \right\}^{\frac{1}{p}} \left\{ \int_t^T \left\| Re^{A(\tau - t)} Bg(t) \right\|_Z^q d\tau \right\}^{\frac{1}{q}} dt \\ \text{(by (7.2.7))} \quad &\leq \| f \|_{L_p(0,T;Z)} \int_0^T C_T \| g(t) \|_U \, dt \\ &= C_T \| f \|_{L_p(0,T;Z)} \| g \|_{L_1(0,T;U)}. \end{aligned} \tag{7.2.9}$$

An obvious variation leads likewise to (7.2.9) also for $p = \infty$. Thus, (7.2.9) says that

$$L^* R^* : \text{ continuous } L_p(0, T; Z) \to L_\infty(0, T; U). \tag{7.2.10}$$

Next, the $L_\infty(0, T; U)$-regularity in (7.2.10) is lifted up to the $C([0, T]; U)$-regularity in (7.2.8b), by an approximation argument, as in Step 2, above.

First, if $f \in H^1(0, T; Z)$, then integration by parts on (7.2.8a) yields only under (H.1) and (H.2):

$$(L^* R^* f)(t) = \int_t^T B^* A^{*-1} \frac{de^{A^*(\tau - t)}}{d\tau} R^* f(\tau)\, d\tau \tag{7.2.11}$$

$$\begin{aligned} &= B^* A^{*-1} e^{A^*(T - t)} R^* f(T) - B^* A^{*-1} R^* f(t) \\ &\quad - \int_t^T B^* A^{*-1} e^{A^*(\tau - t)} R^* \dot{f}(\tau)\, d\tau \in C([0, T]; U). \end{aligned} \tag{7.2.12}$$

Next, let $f \in L_p(0, T; Z)$. By taking now $f_n$ smooth, say $f_n \in C^1([0, T]; Z)$, with $f_n \to f$ in $L_p(0, T; Z)$, we then see that $L^* R^* f_n \in C([0, T]; U)$ by (7.2.12), whereas $L^* R^* f_n \to L^* R^* f$ in $L_\infty(0, T; Z)$ by (7.2.10), and thus $L^* R^* f \in C([0, T]; Z)$, as desired.

## 7.3 A Lifting Regularity Property When $e^{At}$ Is a Group

Under assumptions (H.1) and (H.2) and, moreover, with $e^{At}$ a group, the next result lifts the time regularity from $L_p$ to $C$, while preserving the same space regularity. For the classes of PDEs, we have in mind, and for which the original hypothesis (7.3.1) applies, the assumption that $e^{At}$ is a group is automatically satisfied. The result is false for general semigroups; see Remark 7.3.1.

***Theorem 7.3.1*** *Assume (H.1) and (H.2), and, moreover, that $G(t) = e^{At}$ is a s.c. group. Furthermore, suppose that the operator $L$ in (7.1.1) satisfies*

$$L : \textit{continuous } L_p(0, T; U) \to L_p(0, T; X), \quad 1 \le p < \infty. \tag{7.3.1}$$

*Then, in fact, property (H.3) = (7.1.5) holds true, and then*

$$L : \textit{continuous } L_p(0, T; U) \to C([0, T]; X). \tag{7.3.2}$$

*Proof.* By duality on (7.3.1) we obtain that the operator $L^*$ in (7.2.2a) satisfies, with $1/p + 1/q = 1$:

$$L^* : \text{continuous } L_q(0, T; X^*) \to L_q(0, T; U^*). \tag{7.3.3}$$

We now use the group assumption on $e^{At}$ and apply $L^*$ to a *special* function, given by the dual free ($B \equiv 0$) dynamics of (7.1.3), backward in time, that is, to

$$\hat{f}(\tau) = e^{A^*(-\tau)} x^* \in L_q(0, T; X^*), \quad x^* \in X^*. \tag{7.3.4}$$

Then, from (7.2.2), we obtain

$$\begin{aligned}(L^* \hat{f})(t) &= \int_t^T B^* e^{A^*(\tau - t)} e^{A^*(-\tau)} x^* \, d\tau \\ &= (T - t) B^* e^{A^*(-t)} x^* \in L_q(0, T; U^*),\end{aligned} \tag{7.3.5}$$

recalling (7.3.3), and hence $B^* e^{A^*(-t)} x^* \in L_q(0, T - \epsilon; U^*)$, for any $0 < \epsilon < T$. Since this conclusion holds true for any finite $T$ and any $\epsilon > 0$ small (recall Remark 7.1.1), we then obtain that

$$B^* e^{A^*(-t)} x^* \in L_q(0, T; U^*), \quad x^* \in X^*, \tag{7.3.6}$$

continuously; that is, writing $x^* = e^{A^* T} y^*$, or $y^* = e^{A^*(-T)} x^* \in X^*$, we have from (7.3.6)

$$B^* e^{A^*(T-t)} \ : \ \text{continuous } X^* \to L_q(0, T; U^*), \tag{7.3.7}$$

which is (H.3) = (7.1.5c), as desired. The equivalence between (H.3) and (7.3.3) noted in Theorem 7.2.1(i) completes the proof. □

***Remark 7.3.1*** The lifting regularity result of Theorem 7.3.1 can be applied to mixed problems for second-order hyperbolic equations, for Euler–Bernoulli equations, for Kirchoff equations, etc.; see Lasiecka and Triggiani [1983; 1991]. A few applications will be made in Chapter 9, Section 9.8 in the study of the regularity of wave and Kirchoff equations with interior point control.

***Remark 7.3.2*** The assumption that $e^{At}$ be a s.c. *group* is crucial in the above theorem, in the sense that if $e^{At}$ is only a s.c. semigroup even if holomorphic, the conclusion of the theorem is false. As an example illustrating this, let $y(t, x)$ be the solution of a corresponding *parabolic* equation with, say, zero initial condition and with forcing term $u$ in the Dirichlet boundary conditions,

$$\begin{cases} y_t = \Delta y, & \text{in } (0, T] \times \Omega = Q_T; \\ y(0, \cdot) = 0, & \text{in } \Omega; \\ y|_{\Sigma_T} = u, & \text{in } (0, T] \times \Gamma = \Sigma_T, \end{cases} \tag{7.3.8}$$

as in Chapter 3. Then the corresponding free solution ($u \equiv 0$) is described by a s.c., holomorphic semigroup on $L_2(\Omega)$ (which therefore is not a group). We have that the map $u \to y$ is continuous from $L_2(\Sigma_T) \to L_2(Q_T)$ (even $L_2(\Sigma_T) \to L_2(0, T; H^{\frac{1}{2}}(\Omega))$, see Chapter 3, Section 3.1, yet the map $u \to y(T)$ from $L_2(\Sigma_T)$ to $L_2(\Omega)$ is *not* continuous. In fact, for a preassigned $0 < T < \infty$, one may construct $u \in L_2(\Sigma_T)$ whose corresponding solution $y$ satisfies $y(T) \notin L_2(\Omega)$ [Lions, 1971, p. 202], even in the one-dimensional case.

***Remark 7.3.3*** The above proof extends almost verbatim to other settings as well. For instance, one may replace assumption (7.3.1) by

$$L : \text{continuous } H^1(0, T; U) \to L_2(0, T; X), \tag{7.3.9}$$

$U = U^*$, $X = X^*$ (Hilbert spaces), and then the conclusion (7.3.2) becomes

$$L : \text{continuous } H^1(0, T; U) \to C([0, T]; X). \tag{7.3.10}$$

We shall not need these settings, however.

*Proof of implication. (7.3.9) ⇒ (7.3.10).* Under assumption (7.3.9), the counterparts of (7.3.6) and (7.3.7) are now

$$B^* e^{A^*(\cdot - t)} x \in [H^1(0, T; U)]' \tag{7.3.11}$$

continuously in $x \in X$, and

$$B^* e^{A^*(T-t)} : \text{continuous } X \to [H^1(0, T; U)]', \tag{7.3.12}$$

respectively, where $[H^1(0, T; U)]'$ is the dual of $H^1(0, T; U)$ with respect to

$L_2(0, T; U)$ as a pivot space. From here, it follows that

$$L_T u = \int_0^T e^{A(T-t)} Bu(t)\, dt : \text{continuous } H^1(0, T; U) \to X. \tag{7.3.13}$$

Finally, with $v \in L_1(0, T; X)$ and $u \in H^1(0, T; U)$, one obtains

$$\int_0^T ((Lu)(t), v(t))_X\, dt = \int_0^T \left( \int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau, v(t) \right) dt$$

$$= \int_0^T \int_0^t (u(\tau), B^* e^{A^*(t-\tau)} v(t))_U\, d\tau dt \tag{7.3.14}$$

$$\leq \int_0^T \|u\|_{H^1(0,t;U)} \|B^* e^{A^*(t-\cdot)} v(t)\|_{[H^1(0,t;U)]'}\, dt \tag{7.3.15}$$

$$\text{(by (7.3.12))} \quad \leq \int_0^T \|u\|_{H^1(0,T;U)} c_t \|v(t)\|_X\, dt \tag{7.3.16}$$

$$\leq c_T \|u\|_{H^1(0,T;U)} \int_0^T \|v(t)\|_X\, dt, \tag{7.3.17}$$

as desired, which is the counterpart of (7.2.5). In going from (7.3.15) to (7.3.16), we have recalled (7.3.12) and used that $c_t \leq c_T$ for all $t \leq T$, by duality on the original assumption (7.3.10). Then (7.3.17) yields conclusion (7.3.10) as desired, by an approximation argument, as in Step 2 in the proof of Theorem 7.2.1.

## 7.4 Extension of Regularity of $L$ and $L^*$ on $[0, \infty]$ When $e^{At}$ Is Uniformly Stable

In this section, in addition to (H.1), (H.2), and (H.3), we assume the following "uniform exponential stability" property of $e^{At}$, that is, stability in the uniform operator topology:

(H.4) There exist constants $M \geq 1$, $\omega > 0$, such that

$$\|e^{At}\|_{\mathcal{L}(X)} \leq M e^{-\omega t}, \quad t \geq 0. \tag{7.4.1}$$

As a consequence of (H.4), we extend the continuity of the operator $L$ in (7.1.1) from $T < \infty$ to $T = \infty$. We shall provide a direct statement and a direct proof in Section 7.4.1 and a dual statement and a dual proof in Section 7.4.2.

### *7.4.1 Direct Statement; Direct Proof*

***Theorem 7.4.1.1*** *Assume (H.1), (H.2), (H.3) = (7.1.5), and (H.4) = (7.4.1). Let* $1 \leq p < \infty$. *Let* $\epsilon > 0$ *be such that*

$$-\omega + \epsilon < 0, \tag{7.4.1.1}$$

*with $\omega$ the constant in (7.4.1). Then hypothesis (H.3), in its version given by (7.2.1), that is, $L$: continuous $L_p(0, T; U) \to C([0, T]; X)$, $1 \le p < \infty$, can be improved to the following statements:*

$$e^{\epsilon t} L \ : \ \text{continuous } L_p(0, \infty; U) \to L_p(0, \infty; X) \tag{7.4.1.2}$$

$$: \ \text{continuous } L_p(0, \infty; U) \to C([0, \infty]; X), \tag{7.4.1.3}$$

*where the latter is the space of $X$-valued continuous functions bounded on $[0, \infty]$, that is, bounded under the sup norm.*

*Proof. Proof of (7.4.1.2).* We first prove (7.4.1.2) for $\epsilon = 0$. We let $f(t) = \|(Lu)(t)\|_X^p$ for $u \in L_p(0, \infty; U)$ and obtain

$$\int_0^\infty \|(Lu)(t)\|_X^p \, dt = \int_0^\infty f(t)\, dt = \sum_{n=0}^\infty \int_{nT}^{(n+1)T} f(t)\, dt$$
$$= \sum_{n=0}^\infty \int_0^T f(nT + t)\, dt. \tag{7.4.1.4}$$

By splitting the interval $[0, nT + t]$ in $[0, T]$, $[T, 2T]$, etc., and with a change of variable, we compute for $n = 1, 2, \dots,$ via (7.1.1) and the semigroup property

$$f(nT + t) = \|(Lu)(nT + t)\|_X^p = \left\| \int_0^{nT+t} e^{A(nT+t-\tau)} Bu(\tau)\, d\tau \right\|_X^p$$
$$= \left\| \sum_{j=1}^n e^{A((n-j)T+t)} \int_0^T e^{A(T-\tau)} Bu((j-1)T + \tau)\, d\tau \right.$$
$$\left. + \int_0^t e^{A(t-\tau)} Bu(nT + \tau)\, d\tau \right\|_X^p. \tag{7.4.1.5}$$

Using the assumed stability (7.4.1) = (H.4) and the continuity (7.2.1b), which is equivalent to (H.3), we get

$$f(nT + t) \le k_T^p \left\{ Me^{-\omega t} \sum_{j=1}^n e^{-\omega(n-j)T} \left( \int_0^T \|u((j-1)T + \tau)\|_U^p \, d\tau \right)^{\frac{1}{p}} \right.$$
$$\left. + \left( \int_0^T \|u(nT + \tau)\|^p \, d\tau \right)^{\frac{1}{p}} \right\}^p$$
$$\le k_T^p c_p \left\{ M^p e^{-\omega p t} \left( \sum_{j=1}^n e^{-\omega(n-j)T} \|u\|_{L_p((j-1)T, jT; U)} \right)^p \right.$$
$$\left. + \|u\|^p_{L_p(nT, (n+1)T; U)} \right\}, \tag{7.4.1.6}$$