Automorphic Forms on $\mathrm{SL}_2(\mathbb{R})$ provides an introduction to some aspects of the analytic theory of automorphic forms on $G = \mathrm{SL}_2(\mathbb{R})$ or the upper half-plane X, with respect to a discrete subgroup Γ of G of finite covolume. The point of view is inspired by, but does not assume knowledge of, the theory of infinite dimensional unitary representations of G – until the last sections, whose purpose is to introduce this theory and relate it to automorphic forms.

The topics treated include the construction of fundamental domains, the notion of automorphic form on $\Gamma\backslash G$, its relationship with the classical automorphic forms on X, Poincaré series, constant terms, cusp forms, finite dimensionality of the space of automorphic forms of a given type, compactness of certain convolution operators, Eisenstein series, their analytic continuation, unitary representations of G, and the spectral decomposition of $L_2(\Gamma\backslash G)$.

The main prerequisites are some results in functional analysis (reviewed, with references) and some familiarity with the elementary theory of Lie groups and Lie algebras, used only for G and its analytic subgroups.

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TO GABY

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Preface

A more accurate title would be: *Introduction to some aspects of the analytic theory of automorphic forms on* $\mathrm{SL}_2(\mathbb{R})$ *and the upper half-plane* X. Originally, automorphic forms were holomorphic or meromorphic functions on X satisfying certain conditions with respect to a discrete group Γ of automorphisms of X, usually with fundamental domain of finite (hyperbolic) area. Later on, H. Maass – and then A. Selberg and W. Roelcke – dropped the assumption of holomorphicity, requiring instead that the functions under consideration be eigenfunctions of the Laplace–Beltrami operator. In the 1950s it was realized (in more general cases) – initially by I. M. Gelfand and S. V. Fomin, and then by Harish-Chandra – that the automorphic forms (holomorphic or not) could be equivalently viewed as functions on $\Gamma\backslash\mathrm{SL}_2(\mathbb{R})$ satisfying certain conditions familiar in the theory of infinite dimensional representations of semisimple Lie groups. This led to a new outlook, where the Laplace–Beltrami operator is replaced by the Casimir operator and the theory of automorphic forms becomes closely related to harmonic analysis on $\Gamma\backslash\mathrm{SL}_2(\mathbb{R})$. This is the point of view adopted in this presentation. However, in order to limit the prerequisites, no knowledge of representation theory is assumed until the last sections, a main purpose of which is precisely to make this connection explicit. A fundamental role is played throughout by a theorem stating that a function on $\mathrm{SL}_2(\mathbb{R})$ satisfying certain assumptions ($\mathcal{Z}$-finite and K-finite) is fixed under convolution by some smooth function with arbitrarily small compact support around the identity element (2.14). This is indeed best understood in the context of representation theory (and valid for a general semisimple group), but the proof in the case of $\mathrm{SL}_2(\mathbb{R})$ can and will be described without reference to the general theory.

This topic has now been investigated for well over one hundred years from a variety of angles, and is so rich that there is comparatively not that much overlap between various treatments. Here, we rather single-mindedly explore one direction. As a compensation of sorts, we have tried in Section 0 to give a short

(admittedly incomplete) historical introduction that starts from some other more classical topics, indicating how the present point of view arose and how the two are related. However, Section 0 is not referred to later (and may therefore be skipped): whatever from it is used later will be introduced anew. The Introduction ends with a short description of the contents of the various sections; Section 1 lists the prerequisites.

This book is an outgrowth of the notes for a short course given at the Mathematical Institute of the Academia Sinica in spring 1993, upon an invitation of Dr. Ki Keng Lu of the Academia Sinica. A first draft was written by Dr. Xuning Feng, completed by me, translated into Chinese by Dr. Feng and published in the Chinese periodical *Advances in Mathematics* [8]. I have kept much of it, expanding it to more than twice its original size. These notes were the catalytic agent for the present book, but only its immediate predecessor; they came after several series of lectures and uncompleted attempts to write an exposition. A first introduction to the general case was given at the University of Paris in spring 1964, informally published in the last part of [5]. Later I gave courses on automorphic forms on $SL_2(\mathbb{R})$ at MIT in the fall of 1969, at the Universidad Naçional Autonoma in Mexico City in summer 1979, and on automorphic forms on general reductive groups at Yale (fall 1978) and at the Tata Institute of Fundamental Research, Bombay (January–March 1983; unpublished notes by T. N. Venkataramana). Moreover, around 1984, D. Husemoller and I embarked on a project aiming at an exposition of the one-variable case. A certain number of sections were drafted, but this was eventually dropped. In writing up this final version, I have benefited from these various attempts and thank the auditors at these lectures, and D. Husemoller, for their patience and help.

I would also like to thank H. Jacquet for some useful discussions on the later part of the book and a simplification, as well as P. Sarnak, who encouraged me to complete the notes of my Beijing course and offered to have the outcome published in this series. I am grateful to Stephen D. Miller for a very careful reading of the text, thanks to which many typos, minor blemishes, and a blunder have been caught. Thanks are also due to Elly Gustafsson, who speedily and skillfully converted into beautiful AMS-TEX an endless stream of rather unattractive mixtures of handwritten and typewritten drafts.