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Ergodicity for Infinite Dimensional Systems

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Preface

This book is devoted to asymptotic properties of solutions of stochastic evolution equations in infinite dimensional spaces. It is divided into three parts: Markovian Dynamical Systems, Invariant Measures for Stochastic Evolution Equations, and Invariant Measures for Specific Models.

In the first part of the book we recall basic concepts of the theory of dynamical systems and we link them with the theory of Markov processes. In this way such notions as ergodic, mixing, strongly mixing Markov processes will be special cases of well known concepts of a more general theory. We also give a proof of the Koopman–von Neumann ergodic theorem and, following Doob, we apply it in Chapter 4 to a class of regular Markov processes important in applications. We also include the Krylov–Bogoliubov theorem on existence of invariant measures and give a semigroup characterization of ergodic and mixing measures.

The second part of the book is concerned with invariant measures for important classes of stochastic evolution equations. The main aim is to formulate sufficient conditions for existence and uniqueness of invariant measures in terms of the coefficients of the equations.

We develop first two methods for establishing existence of invariant measures exploiting either compactness or dissipativity properties of the drift part of the equation. We also give necessary and sufficient conditions for existence of invariant measures for general linear systems. We do not discuss infinite dimensional versions of classical methods based on Harris and Doeblin's conditions and on embedded Markov chains mainly because at the moment they do not have too many applications, see however S. Jacquot and G. Royer [93]. Liapunov type techniques are an object of G. Leha and G. Ritter [105], and L. Stettner [147].

Uniqueness of invariant measures is deduced either by a dissipativity argument or by using structural properties of Markov processes like the strong Feller property and irreducibility. Applying Doob's theorem we give sufficient conditions for convergence of transition probabilities to the unique invariant measure. In the final chapter of the second part the regularity of invariant measures is discussed. General conditions are given under which invariant measures are absolutely continuous with respect to a properly chosen Gaussian reference measure. It is shown that under additional requirements the density of the invariant measure belongs to suitably defined Sobolev spaces. We also examine the so–called gradient systems for which explicit formulae for the densities exist. For a different way of investigating uniqueness of invariant measures based on the idea of coupling we refer to C. Mueller [120], where a specific case is treated.

Methods and results developed in the first two parts are applied to specific models in Part III. In many instances the general theory had to be modified to cover interesting cases. Existence and uniqueness of invariant measures are discussed first for various classes of Ornstein–Uhlenbeck processes including wave equations, some equations of financial mathematics and processes in random environments. Next, two chapters are devoted respectively to delay equations and to reaction–diffusion equations in both bounded and unbounded domains. Then invariant measures for classical and spin systems are discussed by the dissipativity method. In particular, exponential convergence of transition probabilities to equilibrium is established.

The final two chapters are devoted to stochastic equations of fluid dynamics: Burgers and Navier–Stokes equations. Asymptotic analysis required here rather sophisticated (technically) considerations.

The majority of the results presented in this book is based on recent results by the authors and their collaborators. Theorems on genuinely dissipative and delay equations and on systems perturbed through the boundary as well as the direct proof of the existence of a solution to the stochastic Navier–Stokes equation appear here for the first time in printed form.

Motivated by applications we have discussed only the so–called mild solutions of evolution equations perturbed by the Wiener process. We have not investigated asymptotic properties of more general martingale solutions and we refer to M. Viot [156], [154], [155], D. Gątarek and B. Goldys [77], and F. Flandoli and B. Maslowski [67], for results in this direction.

A complete description of all invariant measures for linear evolution equations with the noise process being a homogeneous process with independent increments is given in A. Chojnowska–Michalik [25], see also J. Zabczyk [167], for an earlier work on the finite dimensional case, and V. I Bogach ev, M. Röckner and B. Schmuland [12], for connections with the general Dirichlet spaces.

Existence and uniqueness of invariant measures on Polish spaces are discussed in A. G. Bhatt and R. L. Karandikar [9], see also P. E. Echeveria [60] and S. N. Ethier and T. G. Kurz [63], by using the concept of characteristic operators (generators) of the Markov process. This general approach is, however, not applicable to the examples studied in Part III of the book. On the other hand characteristic operators are used in Chapter 8 of the book devoted to the densities of invariant measures.

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