Part I

Markovian Dynamical Systems

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Chapter 1

General Dynamical Systems

In this chapter we recall basic concepts from the theory of dynamical systems which will play an important role in the sequel. We also state Birkhoff's ergodic theorem and give a proof of the Koopman–von Neumann theorem on weakly mixing systems.

1.1 Basic concepts

Let $S = (\Omega, \mathcal{G}, \mathbb{P}, \theta_t)$ be a *dynamical system* consisting of a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ and a group of invertible, measurable transformations θ_t, $t \in \mathbb{R}$, from Ω into Ω, preserving measure $\mathbb{P}$:

$$\mathbb{P}(\theta_t A) = \mathbb{P}(A), \text{ for arbitrary } A \in \mathcal{G} \text{ and } t \in \mathbb{R}. \tag{1.1.1}$$

The group θ_t, $t \in \mathbb{R}$, induces a group of linear transformations U_t, $t \in \mathbb{R}$, either on the real Hilbert space $\mathcal{H} = L^2(\Omega, \mathcal{G}, \mathbb{P})$ or on the complex Hilbert space $\mathcal{H}_{\mathbb{C}} = L^2_{\mathbb{C}}(\Omega, \mathcal{G}, \mathbb{P})$, by the formula

$$U_t\xi(\omega) = \xi(\theta_t\omega),\ \xi \in \mathcal{H} \ (\text{resp. } (\mathcal{H}_{\mathbb{C}})),\ \omega \in \Omega,\ t \in \mathbb{R}. \tag{1.1.2}$$

We shall denote by $\langle \cdot, \cdot \rangle$ the scalar product in $\mathcal{H}$ (resp. $\mathcal{H}_{\mathbb{C}}$). It is clear that the operators U_t, $t \in \mathbb{R}$, are unitary and that $U_t^* = U_{-t}$, since, by the invariance of $\mathbb{P}$,

$$\begin{aligned}\langle U_t\xi,\eta\rangle &= \int_\Omega \xi(\theta_t\omega)\eta(\omega)\,\mathbb{P}(d\omega)\\ &= \int_\Omega \xi(\omega)\eta(\theta_{-t}\omega)\,\mathbb{P}(d\omega)\\ &= \langle \xi, U_{-t}\eta\rangle,\ \xi\in\mathcal{H}\ \ (\text{resp. } \mathcal{H}_{\mathbb{C}}),\ t\in\mathbb{R}.\end{aligned}$$

A dynamical system $S=(\Omega,\mathcal{G},\mathbb{P},\theta_t)$ is said to be *continuous* if

$$\lim_{t\to 0} U_t\xi = \xi \text{ for arbitrary } \xi\in\mathcal{H}\ \ (\text{resp. } \mathcal{H}_{\mathbb{C}}). \tag{1.1.3}$$

We will restrict our considerations only to continuous systems S.

A dynamical system $S=(\Omega,\mathcal{G},\mathbb{P},\theta_t)$ is called *ergodic* if

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T \mathbb{P}(\theta_{-t}A\cap B)dt = \mathbb{P}(A)\mathbb{P}(B), \text{ for all } A,B\in\mathcal{G}. \tag{1.1.4}$$

Equivalently, in terms of the group U_t, $t\in\mathbb{R}$, a system S is ergodic if

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T \langle U_t\xi,\eta\rangle dt = \langle\xi,1\rangle\langle 1,\eta\rangle, \text{ for all } \xi,\eta\in\mathcal{H}_{\mathbb{C}}. \tag{1.1.5}$$

In fact (1.1.5) with $\xi=\chi_A$ and $\eta=\chi_B$ implies (1.1.4). Conversely from (1.1.4) it follows that (1.1.5) holds when ξ and η are simple, and so for all $\xi,\eta\in\mathcal{H}_{\mathbb{C}}$.

A dynamical system $S=(\Omega,\mathcal{G},\mathbb{P},\theta_t)$ is called *weakly mixing* if there exists a set $I\subset[0,+\infty[$ of relative measure 1 such that

$$\lim_{t\to+\infty, t\in I}\mathbb{P}(\theta_{-t}A\cap B) = \mathbb{P}(A)\mathbb{P}(B), \text{ for all } A,B\in\mathcal{G}. \tag{1.1.6}$$

A set $I\subset[0,+\infty[$ is said to have relative measure 1 if

$$\lim_{T\to\infty}\frac{1}{T}\ell_1(I\cap[0,T]) = 1, \tag{1.1.7}$$

where ℓ_1 denotes the Lebesgue measure on $\mathbb{R}$. Equivalently, a system S is weakly mixing if for a set $I\subset[0,+\infty[$ of relative measure 1

$$\lim_{t\to+\infty, t\in I}\langle U_t\xi,\eta\rangle = \langle\xi,1\rangle\langle 1,\eta\rangle, \text{ for all } \xi,\eta\in\mathcal{H}_{\mathbb{C}}. \tag{1.1.8}$$

Finally, a system S is said to be *strongly mixing* if

$$\lim_{t\to+\infty} \mathbb{P}(\theta_{-t}A \cap B) = \mathbb{P}(A)\mathbb{P}(B), \text{ for all } A, B \in \mathcal{G}, \tag{1.1.9}$$

or equivalently, if

$$\lim_{t\to+\infty} \langle U_t\xi, \eta\rangle = \langle \xi, 1\rangle\langle 1, \eta\rangle, \text{ for all } \xi, \eta \in \mathcal{H}_{\mathbb{C}}. \tag{1.1.10}$$

It is clear that a strongly mixing system is weakly mixing and that a weakly mixing system is ergodic.

Remark 1.1.1 For a thorough discussion and motivation of the concepts introduced we refer to K. Petersen [125]. See in particular Chapter 2.

1.2 Ergodic Systems and the Koopman–von Neumann Theorem

Since for any (continuous) dynamical system S the corresponding group U_t, $t \in \mathbb{R}$, is a C_0–group of unitary transformations on $\mathcal{H}_{\mathbb{C}}$, therefore, by Stone's theorem, see e.g. M. Reed and B. Simon [127, page 274] the infinitesimal generator of U_t, $t \in \mathbb{R}$, is of the form $i\mathcal{A}$ where $\mathcal{A}$ is a self-adjoint operator acting on $\mathcal{H}_{\mathbb{C}}$. $\mathcal{A}$ is called the *infinitesimal generator* of S. We will need the following characterization of ergodic and weakly mixing systems:

Theorem 1.2.1 *Let S be a continuous dynamical system, and let $\mathcal{A}$ be its infinitesimal generator.*

(i) S is ergodic if and only if 0 is a simple eigenvalue of $\mathcal{A}$.

(ii) S is weakly mixing if and only if the operator $\mathcal{A}$ has no eigenvalues $\lambda \neq 0$ and 0 is a simple eigenvalue of $\mathcal{A}$.

The characterization of weakly mixing systems given in (ii) is called the Koopman–von Neumann theorem.

Remark 1.2.2 Since $U_t 1 = 1$ for $t \in \mathbb{R}$, 0 is a simple eigenvalue of $\mathcal{A}$ if and only if the only elements $\xi \in \mathcal{H}_{\mathbb{C}}$ (resp. $\mathcal{H}$) such that

$$U_t\xi = \xi \text{ for all } t \in \mathbb{R}, \tag{1.2.1}$$

are constant functions.

In a similar manner the operator $\mathcal{A}$ has no eigenvalues $\lambda \neq 0$ and 0 is a simple eigenvalue of $\mathcal{A}$, if and only if from the identity

$$U_t\xi = e^{i\lambda t}\xi \tag{1.2.2}$$

valid for some $\lambda \in \mathbb{R}, \xi \in \mathcal{H}_{\mathbb{C}}$ and all $t \in \mathbb{R}$ it follows that $\lambda = 0$ and ξ is a constant function. ∎

To prove Theorem 1.2.1 we will take for granted Birkhoff's ergodic theorem, whose proof can be found in any textbook on ergodic theory, see e.g. K. Petersen [125, Theorem 2.3].

Theorem 1.2.3 *Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space, $\Theta : \Omega \to \Omega$ a measure preserving transformation and $\xi \in \mathcal{H}_{\mathbb{C}}$. Then for all $\xi \in \mathcal{H}_{\mathbb{C}}$ there exists $\xi^* \in \mathcal{H}_{\mathbb{C}}$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \xi(\Theta^k(\omega)) = \xi^*(\omega), \omega \in \Omega, \tag{1.2.3}$$

$\mathbb{P}$–a.s. and in $\mathcal{H}_{\mathbb{C}}$.
Moreover

$$\xi^*(\omega) = \xi^*(\Theta(\omega)), \text{ for } \mathbb{P}\text{–a.s. } \omega \in \Omega, \tag{1.2.4}$$

and

$$\mathbb{E}\xi = \mathbb{E}\xi^*, \tag{1.2.5}$$

where $\mathbb{E}\xi = \int_\Omega \xi(\omega)\mathbb{P}(d\omega)$ denotes the expectation of ξ.

Proof of Theorem 1.2.1 — (i) Assume that 0 is a simple eigenvalue of $\mathcal{A}$. We will show that (1.1.5) holds. Without any loss of generality we can assume that $\xi \geq 0$, $\mathbb{P}$–a.s. For an arbitrary positive number h define

$$\xi_h = \int_0^h U_s\xi ds, \ \ \xi \in \mathcal{H}_{\mathbb{C}}, \tag{1.2.6}$$

and consider θ_h, a fixed measure preserving transformation on Ω. Then

$$\frac{1}{n}\sum_{k=0}^{n-1} \xi_h(\theta_h^k(\omega)) = \frac{1}{n}\int_0^{nh} U_s\xi(\omega)ds, \ \omega \in \Omega,$$

and therefore, by Theorem 1.2.3, for arbitrary $h > 0$ there exists $\xi_h^* \in \mathcal{H}_\mathbb{C}$ such that

$$\lim_{n\to\infty} \frac{1}{n}\int_0^{nh} U_s\xi ds = \xi_h^* \text{ in } \mathcal{H}_\mathbb{C} \text{ and } \mathbb{P}\text{--}qeda.s. \tag{1.2.7}$$

For arbitrary $T \geq 0$ let $n_T = [T/h]$ be the maximal nonnegative integer less or equal to T/h. Then $n_T h \leq T < (n_T+1)h$ and $\mathbb{P}$–a.s.

$$\begin{aligned}\frac{n_T}{(n_T+1)h}\frac{1}{n_T}\int_0^{n_T h} U_s\xi ds \quad &\leq \quad \frac{1}{T}\int_0^T U_s\xi ds \\ &\leq \quad \frac{n_T+1}{n_T h}\frac{1}{n_T+1}\int_0^{(n_T+1)h} U_s\xi ds.\end{aligned}$$

Consequently, for arbitrary $h > 0$

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T U_s\xi ds = \frac{1}{h}\xi_h^* \text{ in } \mathcal{H}_\mathbb{C}. \tag{1.2.8}$$

In particular it follows that $\xi_h^* = h\xi_1^*$; since, from (1.2.4), $U_h\xi_h^* = \xi_h^*$ this implies $U_h\xi_1^* = \xi_1^*$ for all $h \geq 0$. Therefore ξ_1^* is a constant function equal to $\langle \xi, 1\rangle$. This proves (i) in one direction.

Let now a system S be ergodic and let $U_t\xi = \xi$ for all $t \geq 0$. By (1.1.5)

$$\langle \xi,\eta\rangle = \langle \xi,1\rangle\langle 1,\eta\rangle = \langle\langle \xi,1\rangle 1,\eta\rangle\,,$$

for all $\eta \in \mathcal{H}_\mathbb{C}$, and therefore $\xi = (\langle \xi,1\rangle)1$ is a constant function.

(ii) Assume that the system S is ergodic and that $\mathcal{A}$ has no eigenvalues $\lambda \neq 0$. We recall that this means that the spectral measure $E(\cdot)$ determined by the operator $\mathcal{A}$ has no atoms except at 0. Moreover $E(\{0\}) = E_0$ is a projection operator onto constant functions and $E(\{\lambda\}) = 0$ for all $\lambda \neq 0$. It is enough to prove the result for $\xi,\eta \in \text{Ker } E_0$ and for $\xi = \eta$. Let $\{\xi_n\}$ be a basis of $(\text{Ker } E_0)^\perp$ on $\mathcal{H}_\mathbb{C}$. We will prove that there exists a set I of relative measure 1 such that, for all $n \in \mathbb{N}$,

$$\lim_{t\in I, t\to+\infty} \langle U_t\xi_n, \xi_n\rangle = 0. \tag{1.2.9}$$

Now we compute the integral

$$\frac{1}{2T}\int_{-T}^{T} |\langle U_t\xi,\xi\rangle|^2\, dt.$$

From the equality

$$\langle U_t\xi,\xi\rangle = \int_{-\infty}^{+\infty} e^{i\lambda t}\|E(d\lambda)\xi\|^2,$$

we find

$$\begin{aligned}|\langle U_t\xi,\xi\rangle|^2 &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i(\lambda-\mu)t}\|E(d\lambda)\xi\|^2\|E(d\mu)\xi\|^2,\\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \cos[(\lambda-\mu)t]\|E(d\lambda)\xi\|^2\|E(d\mu)\xi\|^2.\end{aligned}$$

It follows that

$$\frac{1}{2T}\int_{-T}^{T}|\langle U_t\xi,\xi\rangle|^2\,dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sin[(\lambda-\mu)T]}{(\lambda-\mu)T}\|E(d\lambda)\xi\|^2\|E(d\mu)\xi\|^2, \tag{1.2.10}$$

and, since the measure $\|E(\cdot)\xi\|^2$ has no atoms,

$$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}|\langle U_t\xi,\xi\rangle|^2\,dt = 0$$

Define

$$\gamma(t) = \sum_{n=1}^{\infty}\frac{1}{2^n\|\xi_n\|^4}|\langle U_t\xi_n,\xi_n\rangle|^2;$$

then $0\le\gamma(t)\le 1$ for all $t\in\,]-\infty,+\infty[$ and

$$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\gamma(t)dt = 0.$$

This implies that there exists a set I, of relative measure 1, such that

$$\lim_{\substack{|t|\to+\infty\\ t\in I}}\gamma(t) = 0;$$

this yields (1.2.9).

Conversely, assume that a system S is weakly mixing; then it is ergodic and therefore, by (i), 0 is a simple eigenvalue of $\mathcal{A}$. If for some real λ such that $\lambda\neq 0$ there exists $\xi\in\mathcal{H}_{\mathbb{C}}$, $\xi\neq 0$, such that $U_t\xi = e^{i\lambda t}\xi$ for $t\ge 0$, then

$$\langle U_t\xi,\xi\rangle = e^{i\lambda t}|\xi|^2, \tag{1.2.11}$$

and therefore (1.1.8) cannot be true for a set I of relative measure 1. This finishes the proof of Theorem 1.2.1. ∎

Let S be a continuous dynamical system.

(i) A set $A \in \mathcal{G}$ is said to be *invariant* with respect to S if, for arbitrary $t \in \mathbb{R}$,

$$U_t \chi_A = \chi_A, \ \mathbb{P}\text{–a.s.}, \tag{1.2.12}$$

or, equivalently, if for every $t \in \mathbb{R}$

$$\mathbb{P}(\theta_t A \cap A) = \mathbb{P}(A) = \mathbb{P}(\theta_t A). \tag{1.2.13}$$

(ii) A measurable function $\alpha : \Omega \to [0, 2\pi[$ is said to be an *angle variable* for a system S if there exists $\lambda \in \mathbb{R}$ such that for every $t \in \mathbb{R}$,

$$U_t \alpha \equiv \lambda t + \alpha \ (\mathrm{mod}\ 2\pi), \mathbb{P}\text{–a.s.} \tag{1.2.14}$$

As a consequence of Theorem 1.2.1 we derive the following important result.

Theorem 1.2.4 *Let S be a continuous dynamical system.*

(i) S is ergodic if and only if for any invariant set A, either $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$.

(ii) S is weakly mixing if and only if any angle variable is constant and corresponds to $\lambda = 0$.

Proof — (i) Assume that S is an ergodic system and A is an invariant set. By the very definition,

$$\mathbb{P}^2(A) = \lim_{T \to +\infty} \frac{1}{T} \int_0^T \mathbb{P}(\theta_t A \cap A) dt = \mathbb{P}(A),$$

and therefore $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$.

To prove the converse implication assume that for a $\xi \in \mathcal{H}_\mathbb{C}$ and for all $t \in \mathbb{R}$, $U_t \xi = \xi$. Without any loss of generality we can assume that ξ is real valued. If ξ is not a constant function then for a number $\alpha \in \mathbb{R}$ the sets $A = \{\omega : \xi(\omega) > \alpha\}$ and $A^c = \{\omega : \xi(\omega) \leq \alpha\}$ are of positive $\mathbb{P}$–measure. This is a contradiction since the sets A and A^c are invariant. We have in fact, for any $\omega \in \Omega$,

$$U_t\chi_A(\omega) = \chi_A(\theta_t\omega) = \begin{cases} 1 & \text{if } \xi(\theta_t\omega) > \alpha, \\ 0 & \text{if } \xi(\theta_t\omega) \leq \alpha. \end{cases}$$

Since $\xi(\theta_t\omega) = \xi(\omega)$ by hypothesis, we have $U_t\chi_A = \chi_A$ so that A is invariant. This shows that ξ is a constant function.

(ii) Assume that S is weakly mixing and that α is its angle variable corresponding to λ, then, for $\xi = e^{i\alpha}$, we have

$$U_t\xi = e^{i\lambda t}\xi, \quad \text{for all } t \in \mathbb{R}. \tag{1.2.15}$$

By Theorem 1.2.1, $\lambda = 0$ and ξ (and thus α as well) is constant. To show the converse implication one can assume that the system S is ergodic. Suppose that (1.2.15) holds for some $\lambda \in \mathbb{R}$ and $\xi \in \mathcal{H}_{\mathbb{C}}$. Then $U_t|\xi| = |\xi|$, $t \in \mathbb{R}$. Therefore $|\xi|$ is a constant function and we can assume that $|\xi| = 1$. Consequently $\xi = e^{i\alpha}$ where α is a real function with values on $[0, 2\pi[$. From (1.2.15)

$$U_t\alpha \equiv \lambda t + \alpha \ (\text{mod } 2\pi), \ \mathbb{P}\text{–a.s.},$$

and the result follows. ■

Remark 1.2.5 The content of Theorem 1.2.4 can be phrased shortly as follows:

A system S is ergodic if and only if it has only trivial invariant sets.

A system S is weakly mixing if and only if it has only trivial angle variables.

Chapter 2

Canonical Markovian Systems

We introduce here dynamical systems determined by Markovian transition semigroups on Polish spaces and give conditions for their continuity.

2.1 Markovian semigroups

Let us first give some notation. In all this chapter E represents a Polish space with metric ρ and, for any $x_0 \in E, \delta > 0$, $B(x_0, \delta)$ is the ball

$$B(x_0,\delta) = \{x \in E: \ \rho(x, x_0) < \delta\}.$$

We denote by $\mathcal{E} = \mathcal{B}(E)$ the σ–field of all Borel subsets of E, and for any $\Gamma \in \mathcal{E}$, by χ_Γ the characteristic function

$$\chi_\Gamma(x) = \begin{cases} 1 & \text{if } x \in \Gamma, \\ 0 & \text{if } x \in \Gamma^c, \end{cases}$$

where $\Gamma^c = E \backslash \Gamma$.

Moreover $B_b(E)$ (resp. $C_b(E)$, $UC_b(E)$, $\mathrm{Lip}(E)$) is the set of all real (or complex) bounded Borel functions (resp. continuous and bounded functions, uniformly continuous and bounded functions, Lipschitz continuous functions) on E, and $\mathcal{M}_1(E)$ is the set of all probability measures defined on $(E, \mathcal{E})$.