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A Mathematical Introduction to Wavelets

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Preface

In many places in engineering or science or mathematics we are faced with a version of the following archetypical problem. We are given a function $f(t)$ defined for $t \in \mathbb{R}$. Let us imagine that this function describes some real-life phenomenon. To make things mathematically simple let us assume that $f \in L_2(\mathbb{R})$. Our aim (admittedly vague) is to transmit (or store or analyze) this function using some 'finite' device. A good illustration might be that f represents a voice signal and we want to transmit it over the telephone lines or put it on a compact disk. The whole function f is given by the totality of its values at points of $\mathbb{R}$, and this makes it a continuum of points – we can not do much with 'finite' device. So let us suppose that as our background knowledge we have some orthonormal basis $(f_n)_{n\in N}$ in $L_2(\mathbb{R})$. Then we know that we can write

$$f = \sum_{n\in N} a_n f_n$$

where the series converges in $L_2(\mathbb{R})$ and the coefficients are uniquely determined by the formulas $a_n = \langle f, f_n\rangle$ for $n \in N$. Thus instead of transmitting the function f it suffices to transmit the sequence of coefficients $(a_n)_{n\in N}$ and let the recipient sum the series himself. It is still not a finite procedure but it looks better. To make it really finite we have to choose a finite set $A \subset N$ such that $\sum_{n\in A} a_n f_n$ will be 'almost equal' to $\sum_{n\in N} a_n f_n$. But life is not perfect, so we cannot expect to have perfect tranmission every time. This means that the recipient is really forming the sum $\sum_{n\in A} \tilde{a}_n f_n$, where $\tilde{a}_n$ is 'close' to a_n, and has to hope that the result is still 'almost equal' to $\sum_{a\in N} a_n f_n$. But what does 'almost equal' mean? If we are in any real situation we have to decide and base this decision on our experience. Mathematically speaking we need a distance between functions, which very often is provided by some norm, usually different from the hilbertian norm.

This is a very general story and there were and are many ways to deal with various special instances of different aspects of this archetypical problem. Wavelets are just one new tool to deal with this type of problem.

The subject of wavelets appeared in the mid 1980s influenced by ideas from both pure mathematics (harmonic analysis, functional analysis, approximation theory, fractal sets etc.) and applied mathematics (signal processing, mathematical physics etc.). Almost instantanously it became a success story with thousands of papers written by now and wide ranging applications. The reader may learn some history of this subject from [87] or [86] or from introductions and comments in [24], [84] or [85]. I do not want to discuss the history in any detail. Let me simply state that by now wavelets find applications in many areas of mathematics, science or technology. Just to show how diverse are applications of wavelets let me say that *Wavelet Literature Survey* [92] divides its entries into the following categories: acoustics, astronomy, atomic decompositions, $ax+b$ group, Bernoulli–Gaussian processes, chord–arc curves, fractal and Cantor sets, frames, Franklin wavelets, Gabor representations, image processing, irregular sampling, non-orthogonal expansions, numerical algorithms, partial differential equations, seismology, signal processing, splines, theory, wavelet bases, and wavelet transform.

In order to make this discussion a bit more precise let us state here (in the case of one variable only) the definition of a wavelet which (naturally enough) is the main concept discussed in this book.

Definition 2.1 *A wavelet is a function $\Psi(t) \in L_2(\mathbb{R})$ such that the family of functions*

$$\Psi_{j,k} =: 2^{j/2}\Psi(2^j t - k)$$

where j and k are arbitrary integers, is an orthonormal basis in the Hilbert space $L_2(\mathbb{R})$.

If we look at this definition we can guess how wavelets fit into the general scheme depicted above. They provide the orthonormal system that we wanted to have as our 'background knowledge'.

When one looks at the above definition for the first time, one question arises immediately: *do wavelets exist at all?* To this question we will give an affirmative answer many times in this book. We start with some examples in Chapter 1 and will give more in subsequent chapters. Let us assume for the time being that wavelets exist. Then the next question springs to mind: why bother with them?

Well, the first and really the most important answer is that wavelets fit very well into many concrete cases of our archetipical story. In short: *they are useful.* Naturally this answer can be justified only after working with wavelets on some concrete problems.

The second answer is: simply for the fun of it. To be more serious; it is not clear that such a function exists, so when it turns out that it indeed exists we have a mild surprise. Thus we may wish to investigate such functions in some detail, to know how many such strange functions exist, what additional properties they may have etc. Also each such function gives us an orthonormal basis in $L_2(\mathbb{R})$. So it may be interesting to investigate such bases. There is also a natural question of generalizations, e.g. what happens in $\mathbb{R}^d$. All this will be studied in this book and I hope that the reader will be convinced that wavelets are interesting.

Since I am a pure mathematician by profession, education and character this second answer is close to my heart. Thus I study wavelets as a beautiful mathematical idea. It seems to me that it is this beauty and simplicity of the wavelet concept that has attracted so many people. This intrinsic simplicity makes wavelets a convenient framework unifying various earlier methods.

Even from this Preface it should be clear that the literature on wavelets in enormous and is growing at a tremendous rate. Nevertheless when in 1993 I began preparing a Part III (beginning graduate students in mathematics) course on wavelets at Cambridge University, I had difficulties in finding a text appropriate for such students. Out of [24], [85], [27] and many expository and research papers, and out of my own experience I put together a course which, with many additions, I repeated in 1994/95 for senior undergraduates in mathematics at Warsaw University. The present book is an outgrowth of those efforts. My idea was to present the essential mathematical core of theory of wavelets. I decided to concentrate on orthonormal wavelets as the most complete and 'cleanest' part of the theory. My aim was to give detailed constructions of the most important wavelets and present the usefulness of wavelet bases in decomposing functions. All this is done in the framework of function spaces.

So this is a purely mathematical book, although constantly I try to make my calculations as explicit as possible and I concentrate on theoretical questions that should have relevance to applications. But regrettably I discuss no *real* applications.

In other words, I take it for granted that wavelets are useful, and this belief is one of the motivations for studying wavelets, but I explain only their basic mathematical theory. I hope that each reader will find a favorite application for wavelets. I also believe that the knowledge of orthogonal wavelets discussed in this book should make the study of various generalizations much easier.

This book is basically a course of lectures aimed at students of mathematics (maybe even pure mathematics). Thus I start rather slowly but as the book progresses the pace quickens a bit. There are also more than a hundred exercises for the reader to solve.

Let me explain the content and organization of this book in more detail.

Chapters 1–4 discuss one-variable orthonormal wavelets (as defined above). This is the backbone of the mathematical theory. Chapter 1 in a sense presents an overview of the book. Without any general theory we discuss two wavelets: the Haar wavelet and the Strömberg wavelet. They were invented and investigated well before the emergence of the general theory or indeed the notion of wavelet. We present the construction and properties of those wavelets and show some sample theorems about convergence of wavelet expansions. As its title suggests, the aim of this chapter is to convey the general spirit of the book by presenting important but relatively easy and explicit examples. Formally its results are not used later except as a motivation or in examples and exercises. Chapter 2 discusses the general theory. We present and discuss here the concept of multiresolution analysis and the scaling function. Then we describe all wavelets associated with a given multiresolution analysis. We also show how the scaling function can be used to build the multiresolution analysis. Here we also discuss in general terms periodic wavelets. In Chapter 3 we show how the above general theory can be applied in concrete cases. We construct Meyer's wavelets and spline wavelets and discuss in detail their smoothness and decay. This chapter also includes a self-contained introduction to spline functions. We also discuss in this chapter examples of wavelets not associated with any multiresolution analysis. Chapter 4 discusses wavelets with compact support. We present a general approach to constructing compactly supported wavelets and apply this to a construction of smooth, compactly supported wavelets. We also present an 'elementary' construction of a continuous wavelet whose support is $[0, 3]$. These Chapters use only the rudiments of Hilbert spaces and the Fourier transform on the real line. Chapter 5 discusses multivariable generalizations. We discuss briefly a tensor product technique and next generalize to $\mathbb{R}^d$ the concept of multiresolution analysis. Then we discuss the general procedures leading from multiresolution analysis to wavelets in this context. It turns out that in general we need a finite *wavelet set* instead of one wavelet. We exhibit many examples of Haar-like wavelets on $\mathbb{R}^d$, i.e. wavelets Ψ such that $|\Psi(x)|$ is the characteristic function of a set. Then we construct more smooth examples.

These five chapters constitute an introduction to constructions of orthogonal wavelets. The rest of the book deals with wavelet expansions. In Chapter 6 we give a self-contained presentation of the basic theory of L_p spaces and H_1 and BMO. Our main results are the necessary interpolation theorems. In Chapter 7 we introduce unconditional convergence of series in Banach spaces and discuss the concept of unconditional basis. In Chapter 8 we prove that good wavelets provide unconditional bases in $L_p(\mathbb{R}^d)$ and in $H_1(\mathbb{R}^d)$ and we show the equivalence of various wavelet bases. We also give a characterization of those function spaces in terms of wavelet expansions. We also discuss periodized Meyer wavelets since they lead to systems of trigonometric polynomials. We conclude with Chapter 9 where, on $\mathbb{R}$ only, we discuss moduli of continuity and Besov norms and their connections with wavelets.

The obvious prerequisite to reading this book is a sound understanding of real analysis, functions, series of functions etc. A familiarity with Lebesgue integration is very useful, in particular the Lebesgue dominated convergence theorem is used several times. Additionally some elementary knowledge of the Fourier transform and Hilbert spaces are needed. All the facts needed are summarized in Sections A.1 and A.2. To read Chapters 1–4 (with the exception of some portions of Chapter 1) one needs only to know the Fourier transform on $\mathbb{R}$ and be familiar with the Hilbert space $L_2(\mathbb{R})$. To read Chapter 5 one needs to know the Fourier transform on $\mathbb{R}^d$. To read the remaining chapters one needs the concept and very basic properties of Banach spaces. They are summarized in Section A.3. The more advanced facts are carefully presented when needed. In particular Chapter 6 contains a detailed and selfcontained presentation of the necessary parts of the theory of L_p spaces and the theory of H_1 and BMO. The concept of unconditional basis is introduced and presented in some detail in Chapter 7.

As the reader could have guessed anyway, the book is organized into chapters, which are divided into sections. The numbering is decimal by chapter. Displayed formulas (if numbered) are numbered consecutively. There is a single numbering sequence for theorems, propositions, lemmas, corollaries and definitions. Each chapter ends with a section 'Sources and comments' and with exercises. Exercises range from quite easy and routine to rather difficult. The last chapter of the book is an Appendix which contains sections listing results used about Hilbert spaces, Fourier transforms and Banach spaces. It also contains list of symbols and list of spaces. References to a result from the Appendix are easily recognizable by the presence of the letter A and Roman numerals, e.g. A.2–II

Since this is clearly a textbook the Bibliography lists only works actually referred to in the text, mostly in 'Sources and comments'. I have not tried to give full bibliography of the subject and it would be impossible anyway. For readers looking for more information on the mathematical part of wavelet literature I can only offer the following suggestions:

- To check recent volumes of *Mathematical Reviews* or *Zentralblatt für Mathematik*. If possible this should be done using the corresponding computer data base. The search for the keyword *wavelet* will immediately exhibit hundreds of items.
- The book *Wavelet Literature Survey* [92] lists more than 1000 items of the wavelet literature. They cover both the theory and many diverse applications.
- There is a whole series *Wavelet analysis and its applications* published by Academic Press, which contains books on various aspects of wavelets. The book [15] was the first publication in this series.
- There are mathematical journals which regularly publish papers connected with wavelets. Browsing through volumes of *Applied and Computational Harmonic Analysis*, *The Journal of Fourier Analysis and Applications*, *Constructive Approximation* or *Studia Mathematica* should easily yield many interesting papers.

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