1
A small sample

The main point of this introductory chapter is to present two wavelets: the Haar wavelet and the Strömberg wavelet. We do so without using any general theory and without even giving the definition of a wavelet. We present the construction of these wavelets and indicate how they can be used to represent functions from some simple, natural classes. This (as the chapter title indicates) represents a sample of this book. Such an approach is also justified historically. Both these wavelets were well known (without the use of the word 'wavelet') before the emergence of the general theory.

1.1 The Haar wavelet

In this section we will discuss in some detail the most elementary wavelet, called the Haar wavelet.

DEFINITION 1.1 *The Haar wavelet is the function defined on the real line* $\mathbb{R}$ *as*

$$H(t) = \begin{cases} 1 & \text{for } t \in [0, \frac{1}{2}) \\ -1 & \text{for } t \in [\frac{1}{2}, 1] \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

We are interested in the family $\{2^{j/2}H(2^jt-k)\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$. To simplify the notation let us denote $H_{jk}(t) =: 2^{j/2}H(2^jt-k)$. Observe that

$$\operatorname{supp} H_{j,k} = [k2^{-j}, (k+1)2^{-j}]. \tag{1.2}$$

The intervals $[k2^j, (k+1)2^j]$ for $k, j \in \mathbb{Z}$ form the family of dyadic intervals. This family of intervals splits naturally into levels; the j-th level consists of intervals whose length is 2^{-j}. Inside each level distinct dyadic intervals are non-overlapping. The following two properties of dyadic intervals will be useful in further considerations:

(i) either two dyadic intervals do not overlap or one is contained in the other
(ii) if one dyadic interval is strictly contained in the other, then it is contained either in the left half or in the right half of it.

Those observations easily give the following proposition.

Proposition 1.2 *The system* $\{2^{j/2}H(2^j t - k)\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ *is orthonormal in* $L_2(\mathbb{R})$.

Proof Let us look at the scalar products $\langle H_{j,k}, H_{j',k'}\rangle$. We can assume that $j \leq j'$. Using the substitution $u = 2^j - k$ we see that

$$\langle H_{j,k}, H_{j',k'}\rangle = \int_{-\infty}^{\infty} 2^{s/2} H(t) H(2^s t - r) dt \tag{1.3}$$

where $s = j' - j$ and $r = 2^{j'-j}k - k'$. If $j = j'$ and $k = k'$ then the integral in 1.3 clearly equals 1. If $j = j'$ but $k \neq k'$ then $r \neq 0$ so $\operatorname{supp} H(2^s t - r) \cap \operatorname{supp} H(t) = \emptyset$ and the integral in 1.3 is 0. When $j' > j$ then we have either $\operatorname{supp} H(2^s t - r) \cap \operatorname{supp} H(t) = \emptyset$ so the integral is 0 or $\operatorname{supp} H(2^s t - r) \not\subseteq \operatorname{supp} H(t)$. But in this case $H(t)$ is constant on $\operatorname{supp} H(2^s t - r)$, so the integral is also 0 because $\int_{-\infty}^{\infty} H(t)dt = 0$. □

In order to show that $\{2^{j/2}H(2^j t - k)\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ is an orthonormal *basis* in $L_2(\mathbb{R})$ let us consider two families of closed subspaces of $L_2(\mathbb{R})$:

$$S_n = \operatorname{span}\{H_{j,k}\}_{j<n,k\in\mathbb{Z}} \tag{1.4}$$

and

$$L_n = \left\{ \begin{matrix} \text{all functions in } L_2(\mathbb{R}) \text{ constant on all} \\ \text{intervals } [k2^{-n}, (k+1)2^{-n}], \text{ for } k \in \mathbb{Z} \end{matrix} \right\}. \tag{1.5}$$

Both these families have the following properties (we formulate them for the S_n's only, but the same holds for the L_n's).

$$\ldots \subset S_{-1} \subset S_0 \subset S_1 \subset \ldots \tag{1.6}$$

$$f(t) \in S_n \iff f(2t) \in S_{n+1} \tag{1.7}$$

$$f(t) \in S_0 \iff f(t+k) \in S_0 \quad \text{for } k \in \mathbb{Z}. \tag{1.8}$$

Lemma 1.3 *For all* $n \in \mathbb{Z}$ *we have* $L_n = S_n$.

Proof From 1.7 above we see that it suffices to show that $S_0 = L_0$. Since each $H_{j,k}$ for $j < 0$ is constant on any interval $[r, r+1]$ we see that $S_0 \subset L_0$. On the other hand each function from L_0 can be written as $\sum_{r\in\mathbb{Z}} a_r \mathbf{1}_{[r,r+1]}$ so by 1.8 it suffices to show that $\mathbf{1}_{[0,1]} \in S_0$.

To show this let us consider the series

$$\sum_{j<0} 2^{j/2} H_{j,0} = \sum_{j<0} 2^j H(2^j t).$$

Since $\|2^j H(2^j t)\|_2 = 2^{j/2}$ and $j < 0$ this series is absolutely convergent in $L_2(\mathbb{R})$. One can easily see from the definition of $H(t)$ that

$$\sum_{j<0} 2^{j/2} H_{j,0}(t) = 0 \quad \text{for} \quad t \leq 0,$$

$$\sum_{j<0} 2^{j/2} H_{j,0}(t) = \sum_{j<0} 2^j = 1 \quad \text{for } 0 < t < 1$$

and for $2^r < t < 2^{r+1}$ where $r = 0, 1, 2, \ldots$ one has

$$\sum_{j<0} 2^{j/2} H_{j,0}(t) = -2^{-r-1} + \sum_{j=r+2}^{\infty} 2^{-j} = 0.$$

This shows that $S_0 = L_0$ so $L_n = S_n$ for all $n \in \mathbb{Z}$. □

From Proposition 1.2 and Lemma 1.3 and the fact that $\bigcup_{n=-\infty}^{\infty} L_n$ is dense in $L_2(\mathbb{R})$ we get immediately:

Theorem 1.4 *The system* $\{2^{j/2} H(2^j t - k)\}_{j\in\mathbb{Z}, k\in\mathbb{Z}}$ *is an orthonormal basis in* $L_2(\mathbb{R})$.

This means that for a function $f \in L_2(\mathbb{R})$ we have a decomposition

$$f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \langle f, H_{jk} \rangle H_{jk}. \tag{1.9}$$

Since $H \in L_p(\mathbb{R})$ for all p, $1 \leq p \leq \infty$, we can write the right hand side for any $f \in L_p(\mathbb{R})$, $1 \leq p \leq \infty$. For the rest of this section we will investigate the convergence of this series for $f \in L_p(\mathbb{R})$. Later (Section 8.2) we will show that for $1 < p < \infty$ this series converges when arranged in any order. For the time being we will discuss only the most natural order. Thus we will study operators Q_j^μ and P_r defined as

$$Q_j^\mu(f) = \sum_{k\leq\mu} \langle f, H_{jk} \rangle H_{jk} \tag{1.10}$$

Cambridge University Press
0521578949 - A Mathematical Introduction to Wavelets
P. Wojtaszczyk
Excerpt
More information

and

$$P_r(f) = \sum_{j<r} \sum_{k \in \mathbb{Z}} \langle f, H_{jk} \rangle H_{jk}. \tag{1.11}$$

Theorem 1.5 *If $f \in L_p(\mathbb{R})$ with $1 < p < \infty$ or $f \in C_0(\mathbb{R})$, then $\lim_{r\to\infty} P_r(f) = f$ and for each $r \in \mathbb{Z}$ $\lim_{\mu\to\infty} P_r(f) + Q_r^\mu(f) = P_{r+1}(f)$. The convergence is in the norm of the space.*

Proof For $1 < p < \infty$ let us denote by S_n^p and L_n^p the spaces defined by 1.4 and 1.5 where the closure is taken in $L_p(\mathbb{R})$ not in $L_2(\mathbb{R})$. The proof of Lemma 1.3 can be easily modified to show that $L_n^p = S_n^p$ for all $n \in \mathbb{Z}$. Since $P_r(f)$ is an orthogonal projection onto S_r and $S_r = L_r$ we can write a different representation of the operator P_r, namely we have

$$P_r(f) = \sum_{k\in\mathbb{Z}} 2^r \int_{k2^{-r}}^{(k+1)2^{-r}} f(t)dt \cdot \mathbf{1}_{[k2^{-r},(k+1)2^{-r}]}.$$

The validity of this representation follows immediately from the fact that the right hand side of the above equation defines an orthogonal projection onto L_r. Let us also note that $\bigcup_{n\in\mathbb{Z}} L_n$ is dense in $L_p(\mathbb{R})$ for $1 < p < \infty$.

Thus the first claim for $1 < p < \infty$ follows from the fact that the norm of P_r as an operator on $L_p(\mathbb{R})$ equals 1. This is a simple consequence of Hölder's inequality as follows:

$$\begin{aligned}
\|P_r f\|_p &= \left(\sum_{k\in\mathbb{Z}} 2^{rp} \left| \int_{k2^{-r}}^{(k+1)2^{-r}} f(t)dt \right|^p 2^{-r} \right)^{1/p} \\
&\leq \left(\sum_{k\in\mathbb{Z}} 2^{rp} \left(\int_{k2^{-r}}^{(k+1)2^{-r}} |f(t)|^p dt \right) 2^{-r} 2^{-rp/q} \right)^{1/p} \\
&= \left(\int_{-\infty}^{\infty} |f(t)|^p dt \right)^{1/p}.
\end{aligned}$$

If $f \in C_0(\mathbb{R})$ then it is uniformly continuous. Thus given $\varepsilon > 0$ we can find an N such that for $r > N$ and for each $k \in \mathbb{Z}$

$$\sup \left\{ |f(x) - f(y)| \; : \; x, y \in [k2^{-r}, (k+1)2^{-r}] \right\} < \varepsilon.$$

For a given $r > N$ and each $t \in \mathbb{R}$ we fix an integer k such that $t \in [k2^{-r}, (k+1)2^{-r}]$. Then

$$\begin{aligned} |P_r f(t) - f(t)| &= \left| 2^r \int_{k2^{-r}}^{(k+1)2^{-r}} f(s)\, ds - f(t) \right| \\ &= \left| 2^r \int_{k2^{-r}}^{(k+1)2^{-r}} \big(f(s) - f(t)\big)\, ds \right| < \varepsilon. \end{aligned}$$

This implies that

$$\sup_{t\in\mathbb{R}} |P_r(f)(t) - f(t)| \to 0 \quad \text{as } r \to \infty.$$

This proves the first claim of the theorem.

Since for a fixed j the functions $\{H_{jk}\}_{k\in\mathbb{Z}}$ have disjoint supports, we have

$$\begin{aligned} \Big\| \sum_{k\le\mu} \langle f, H_{jk}\rangle\, H_{jk} \Big\|_p &= \left(\textstyle\sum_{k\le\mu} |\langle f, H_{jk}\rangle|^p\, \|H_{jk}\|_p \right)^{1/p} \\ &= \left(\textstyle\sum_{k\le\mu} 2^{j(\frac{1}{2}-\frac{1}{p})}\, |\langle f, H_{jk}\rangle|^p \right)^{1/p}. \end{aligned}$$

This shows that $\lim_{\mu\to\infty} Q_j^\mu(f)$ exists in the norm of the space. Clearly it equals $P_{j+1}(f) - P_j(f)$. This completes the proof of the theorem. □

The Haar wavelet is very well localized. The supports of the functions $\{2^{j/2}H(2^j t - k)\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ are dyadic intervals and are easily understood. The chief drawback is that H is not continuous. This implies that even for $f \in C_0(\mathbb{R})$ the functions $P_r f$ are not continuous. They are step functions.

1.2 The Strömberg wavelet

In this section we discuss the Strömberg wavelet, which is a continuous, piecewise linear function. Its definition is much more involved than the definition of the Haar wavelet.

Let us start by defining some subsets of $\mathbb{R}$. Let us put

$$\begin{aligned} \mathbb{Z}_+ &= \{1, 2, \ldots\} \\ \mathbb{Z}_- &= -\mathbb{Z}_+ \\ A_0 &= \mathbb{Z}_+ \cup \{0\} \cup \tfrac{1}{2}\mathbb{Z}_- \\ A_1 &= A_0 \cup \{\tfrac{1}{2}\}. \end{aligned}$$

Note that in the above definitions we use our standard notation that for a number a and a subset $A \subset \mathbb{R}$ we have $aA = \{ax : x \in A\}$ and $a + A = \{a + x : x \in A\}$.

Given a discrete subset $V \subset \mathbb{R}$ let $\mathcal{S}(V)$ be the space of all functions $f \in L_2(\mathbb{R})$ which are continuous on $\mathbb{R}$ and linear on every interval $I \subset \mathbb{R}$ such that $I \cap V = \emptyset$. It is clear that $\mathcal{S}(V)$ is non-empty if $|V| \geq 3$. Also if $V_1 \subset V_2$ are discrete subsets of $\mathbb{R}$ then $\mathcal{S}(V_1) \subset \mathcal{S}(V_2)$. In particular $\mathcal{S}(A_0) \subset \mathcal{S}(A_1)$ are non-trivial closed subspaces of $L_2(\mathbb{R})$, and it is easy to see that $\mathcal{S}(A_0)$ has codimension 1 in $\mathcal{S}(A_1)$. One can simply write each function $f \in \mathcal{S}(A_1)$ as a sum $f = g + \alpha\Lambda$ where $g \in \mathcal{S}(A_0)$ is defined as $g(r) = f(r)$ for $r \in A_0$ and $\Lambda \in (A_1)$ is defined as

$$\Lambda(r) = \begin{cases} 0 & \text{if } r \in A_0 \\ 1 & \text{if } r = \frac{1}{2}. \end{cases} \tag{1.12}$$

DEFINITION 1.6 *The Strömberg wavelet is a function $S \in \mathcal{S}(A_1)$ such that $\|S\|_2 = 1$ and S is orthogonal to $\mathcal{S}(A_0)$.*

REMARK 1.1. Such a function S is actually defined only up to a unimodular multiplicative constant, but this will not matter.

We will proceed analogously to Section 1.1 and show that the system $\{2^{j/2}S(2^jt - k)\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ is an orthonormal basis in $L_2(\mathbb{R})$. First let us check that this system is orthonormal.

The argument for this is similar to the argument given in the previous section for the orthogonality of $\{2^{j/2}H(2^jt - k)\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$. Let us take two pairs (j, k) and (j', k'). Let us assume that $j \leq j'$. Using the substitution $u = 2^jt - k$ we see that

$$\langle S_{jk}, S_{j'k'}\rangle = 2^{s/2}\int_{-\infty}^{\infty} S(t)S(2^st - r)dt \tag{1.13}$$

where $s = j' - j$ and $r = 2^{j'-j}k - k'$.

If $s \geq 1$ (i.e. $j \neq j'$) then for each $r \in \mathbb{Z}$

$$S(2^st - r) \in \mathcal{S}(\tfrac{1}{2}\mathbb{Z}) \subset \mathcal{S}(A_0)$$

so the integral in 1.13 equals zero. If $s = 0$ (i.e. $j = j'$) and $k \neq k'$ then we can assume that $k' > k$. In this situation $r = k - k' < 0$ so $S(t - r) \in \mathcal{S}(A_1 - r) \subset \mathcal{S}(A_0)$. This means that the integral in 1.13 equals zero, so $\{2^{j/2}S(2^jt - k)\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ is an orthonormal system.

The fact that $\{2^{j/2}S(2^jx - k)\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis follows immediately from the following three facts:

$$\bigcup_{n\in\mathbb{Z}} \mathcal{S}(2^{-n}\mathbb{Z}) \quad \text{is dense in} \quad L_2(\mathbb{R}) \tag{1.14}$$

$$\bigcap_{n\in\mathbb{Z}} \mathcal{S}(2^{-n}\mathbb{Z}) = \{0\} \tag{1.15}$$

and

$$\mathcal{S}(2^{-j-1}\mathbb{Z}) = \mathcal{S}(2^{-j}\mathbb{Z}) \oplus \operatorname{span}\{S_{jk}\}_{k\in\mathbb{Z}}. \tag{1.16}$$

The proof of 1.14 is a routine approximation argument, e.g. we know that continuous functions with compact support are dense in $L_2(\mathbb{R})$ and each such function can be approximated by functions from $\mathcal{S}(2^{-n}\mathbb{Z})$.

To see 1.15 note that if $f \in \mathcal{S}(2^n\mathbb{Z})$ for an integer n then f is linear on the intervals $[-2^n, 0]$ and $[0, 2^n]$, so if $f \in \bigcap_{n\in\mathbb{Z}} \mathcal{S}(2^n\mathbb{Z})$ then f is linear on both half-lines $[0, \infty)$ and $(-\infty, 0]$. For a function in $L_2(\mathbb{R})$ this means that $f = 0$.

One easily checks, using an obvious change of variables, that the map $f \mapsto 2^{j/2} f(2^j x)$ is a unitary map of $L_2(\mathbb{R})$. One also checks that it maps $\mathcal{S}(\frac{1}{2}\mathbb{Z})$ onto $\mathcal{S}(2^{-j-1}\mathbb{Z})$ and $\mathcal{S}(\mathbb{Z})$ onto $\mathcal{S}(2^{-j}\mathbb{Z})$ and $\operatorname{span}\{S_{0,k}\}_{k\in\mathbb{Z}}$ onto $\operatorname{span}\{S_{jk}\}_{k\in\mathbb{Z}}$. Thus 1.16 will follow once we establish

$$\mathcal{S}(\tfrac{1}{2}\mathbb{Z}) = \mathcal{S}(\mathbb{Z}) \oplus \operatorname{span}\{S_{0,k}\}_{k\in\mathbb{Z}}. \tag{1.17}$$

To show 1.17 we will use the translation operator T_N defined by the formula $(T_N f)(x) = f(x - N)$. This is clearly a unitary operator on $L_2(\mathbb{R})$. Applying the translation operator T_N to the definition of S we see that for each $N \in \mathbb{N}$

$$\mathcal{S}(A_1 - N) = \mathcal{S}(A_0 - N) \oplus \operatorname{span}\{S_{0,-N}\}. \tag{1.18}$$

Since $A_0 = A_1 - 1$, we can repeatedly apply 1.18 and obtain that for each N and $r \geq 0$

$$\begin{aligned}
&\mathcal{S}(A_1 - (N - r)) \\
&= \mathcal{S}(A_0 - (N - r)) \oplus \operatorname{span}\{S_{0,-N+r}\} \\
&= \mathcal{S}(A_1 - (N - r + 1)) \oplus \operatorname{span}\{S_{0,-N+r}\} \\
&= \mathcal{S}(A_0 - (N - r + 1)) \oplus \operatorname{span}\{S_{0,-N+r}, S_{0,-N+r-1}\} \\
&\vdots \\
&= \mathcal{S}(A_0 - N) \oplus \operatorname{span}\{S_{0,k}\}_{k=-N}^{-N+r}.
\end{aligned}$$

If we take $N > 0$ and $r = 2N$ we get

$$\mathcal{S}(A_1 + N) = \mathcal{S}(A_0 - N) \oplus \operatorname{span}\{S_{0,k}\}_{k=-N}^{N}.$$

Letting $N \to \infty$ we obtain 1.17.

So we know that $\{2^{j/2}S(2^jt-k)\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ *is an orthonormal basis in* $L_2(\mathbb{R})$.

The definition of S gives no direct clue to what it looks like. Our next task is to compute the function S explicitly. It turns out that although S is supported on the whole real line $\mathbb{R}$, it decays very fast at infinity. We have the following:

Theorem 1.7 *At points of the set* A_1 *the values of the Strömberg wavelet* S *are given by*

$$\begin{aligned} S(k) &= S(1)(\sqrt{3}-2)^{k-1} \quad \textit{for} \ \ k=1,2,3,\ldots \\ S(\tfrac{1}{2}) &= -S(1)(\sqrt{3}+\tfrac{1}{2}) \\ S(0) &= S(1)(2\sqrt{3}-2) \\ S(-\tfrac{k}{2}) &= S(1)(2\sqrt{3}-2)(\sqrt{3}-2)^k \quad \textit{for} \ \ k=1,2,3,\ldots \end{aligned}$$

where $S(1)$ *has to be fixed so that* $\|S\|_2=1$. *Note that since* $S\in \mathcal{S}(A_1)$ *the above values determine* S *completely.*

Let us point out some facts which follow immediately from this Theorem.

(a) The Strömberg wavelet S oscillates; because $\sqrt{3}-2<0$ it changes sign between any two consecutive points of A_1.

(b) The Strömberg wavelet S has exponential decay, i.e. there exist constants C and $\alpha>0$ such that

$$|S(t)|\le C\exp(-\alpha|t|) \tag{1.19}$$

for all $t\in\mathbb{R}$. Clearly $\alpha=-\ln(2-\sqrt{3})\sim 1.316$ works.

(c) The Strömberg wavelet S is in $L_p(\mathbb{R})$ for all $1\le p\le\infty$.

(d) We have $S(-\frac{k}{2})=(10-6\sqrt{3})S(k)$ for $k=1,2,3,\ldots$. This shows that S admits a certain symmetry. This symmetry mirrors the fact that A_1 is twice as dense in $(-\infty,0]$ as in $[0,\infty)$.

Proof of Theorem 1.7. For $\sigma\in A_0$ let the function $\Lambda_\sigma\in\mathcal{S}(A_0)$ be defined as

$$\Lambda_\sigma(t)=\begin{cases} 1 & \text{if } t=\sigma \\ 0 & \text{if } t\in A_0 \quad \text{and } t\neq\sigma \\ \text{linear} & \text{otherwise} \end{cases} \tag{1.20}$$

Such functions are called simple tents in $\mathcal{S}(A_0)$. Clearly $\langle S,\Lambda_\sigma\rangle=0$ for all $\sigma\in A_0$. Since S is piecewise linear, these scalar products can easily be computed explicitly in terms of values of S on A_1. From these calculations we obtain

$$0 = \langle S, \Lambda_\sigma \rangle = S(\sigma - 1) + 4S(\sigma) + S(\sigma + 1) \tag{1.21}$$

for $\sigma = 2, 3, 4, \ldots$, and

$$0 = \langle S, \Lambda_\sigma \rangle = S(\sigma - \tfrac{1}{2}) + 4S(\sigma) + S(\sigma + \tfrac{1}{2}) \tag{1.22}$$

for $\sigma = -\frac{1}{2}, -1, -\frac{3}{2}, -2, \ldots$.

The remaining two tents Λ_0 and Λ_1 are a bit more involved. We obtain

$$0 = \langle S, \Lambda_0 \rangle = 2S(-\tfrac{1}{2}) + 9S(0) + 6S(\tfrac{1}{2}) + S(1) \tag{1.23}$$

and

$$0 = \langle S, \Lambda_1 \rangle = S(0) + 6S(\tfrac{1}{2}) + 13S(1) + 4S(2). \tag{1.24}$$

It is natural to look for the solution of the system of equations 1.21 in the form $S(\sigma) = S(1)q^{\sigma-1}$. When we substitute this into 1.21, all equations give for q the same quadratic equation $1 + 4q + q^2 = 0$. It has two solutions $q = -2 \pm \sqrt{3}$. However, S should be in $L_2(\mathbb{R})$ so we need $|q| < 1$; thus we must take $q = \sqrt{3} - 2$.

Now we try to find the solution of 1.22 in the form $S(-\frac{k}{2}) = S(0)q^k$ for $k = 1, 2, 3, \ldots$ and in the same way we obtain the same $q = \sqrt{3} - 2$. If we substitute the values obtained so far for $S(-\frac{1}{2})$ and $S(2)$ into 1.23 and 1.24 we obtain the system

$$\begin{aligned} 0 &= 2(\sqrt{3} - 2)S(0) + 9S(0) + 6S(\tfrac{1}{2}) + S(1) \\ 0 &= S(0) + 6S(\tfrac{1}{2}) + 13S(1) + (\sqrt{3} - 2)S(1). \end{aligned}$$

Solving this system we obtain

$$\begin{aligned} S(0) &= (2\sqrt{3} - 2)S(1) \\ S(\tfrac{1}{2}) &= -(\sqrt{3} + \tfrac{1}{2})S(1). \end{aligned}$$

Thus we obtain a sequence $(S(\sigma))_{\sigma \in A_1}$ satisfying all the equations 1.21–1.24. This sequence yields a function $S \in \mathcal{S}(A_1)$ which is orthogonal to all simple tents in $\mathcal{S}(A_0)$. Since, as is easily seen, all simple tents in $\mathcal{S}(A_0)$ are linearly dense in $\mathcal{S}(A_0)$, we obtain that S is orthogonal to $\mathcal{S}(A_0)$. This S when properly normalized is the Strömberg wavelet. □

Because $S \in L_p(\mathbb{R})$ for all p, $1 \leq p \leq \infty$, the coefficients with respect to the Strömberg basis, namely $\langle f, S_{jk} \rangle = \int_{-\infty}^{\infty} f(t) S_{jk}(t)\, dt$, can be computed for any function f which is in some $L_p(\mathbb{R})$ for $1 \leq p \leq \infty$. So we can consider the convergence of the series

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \langle f, S_{jk}\rangle \cdot S_{jk} \tag{1.25}$$

for f in various function spaces. We will discuss here only the case $f \in C_0(\mathbb{R})$. Analogously to what we did with the Haar wavelet we start by considering operators P_r defined as

$$P_r f = \sum_{j<r}\sum_{k\in\mathbb{Z}} \langle f, S_{jk}\rangle S_{jk} \tag{1.26}$$

which in this case are orthogonal projections onto $\mathcal{S}(2^{-s}\mathbb{Z})$. We have

Theorem 1.8 *There exists a constant C such that for any $f \in C_0(\mathbb{R}) \cap L_2(\mathbb{R})$ the operator P_r defined by 1.26 satisfies*

$$\|P_r f\|_\infty \leq C\,\|f\|_\infty$$

for all $r \in \mathbb{Z}$.

The technicalities of the proof of Theorem 1.8 are contained in the following two lemmas.

Lemma 1.9 *If g_{abc} is a continuous, piecewise linear function on $[-1,1]$ given by*

$$g_{abc}(x) = \begin{cases} a & x=-1 \\ c & x=0 \\ b & x=1 \\ \textit{linear} & \textit{otherwise} \end{cases} \tag{1.27}$$

then

$$I =: \int_{-1}^{1} |g_{abc}(x)|^2 dx = \frac{a^2+b^2+2c^2+ac+bc}{3}.$$

Proof Simply compute

$$\begin{aligned} I &= \int_{-1}^{0} [(c-a)x+c]^2\,dx + \int_{0}^{1} [(b-c)x+c]^2\,dx \\ &= \frac{(a-c)^2}{3} + c(a-c) + c^2 + \frac{(b-c)^2}{3} + c(b-c) + c^2 \\ &= \frac{a^2+b^2+2c^2+ac+bc}{3}. \end{aligned}$$

□