This is an advanced text for first-year graduate students in physics and engineering taking a standard classical mechanics course. It is the first book to describe the subject in the context of the language and methods of modern nonlinear dynamics.

The organizing principle of the text is integrability vs nonintegrability. Flows in phase space and transformations are introduced early and systematically and are applied throughout the text. The standard integrable problems of elementary physics are analyzed from the standpoint of flows, transformations, and integrability. This approach then allows the author to introduce most of the interesting ideas of modern nonlinear dynamics via the most elementary nonintegrable problems of Newtonian mechanics. The book begins with a history of mechanics from the time of Plato and Aristotle, and ends with comments on the attempt to extend the Newtonian method to fields beyond physics, including economics and social engineering.

This text will be of value to physicists and engineers taking graduate courses in classical mechanics. It will also interest specialists in nonlinear dynamics, mathematicians, engineers, and system theorists.

CLASSICAL MECHANICS

transformations, flows, integrable and chaotic dynamics

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For my mother,
Jeanette Gilliam McCauley,
and my father,
Joseph Lee McCauley (1913–1985).
Ebenso
meiner Lebensgefährtin und Wanderkameradin,
Cornelia Maria Erika Küffner.

Middels klok
bør mann være,
ikke altfor klok;
fagreste livet
lever den mennen
som vet måtelig mye.

Middels klok
bør en mann være,
ikke altfor klok;
sorgløst er hjertet
sjelden i brystet
hos ham som er helt klok.

Håvamål

Contents

Foreword

Invariance principles and integrability (or lack of it) are the organizing principles of this text. Chaos, fractals, and strange attractors occur in different nonintegrable Newtonian dynamical systems. We lead the reader systematically into modern nonlinear dynamics via standard examples from mechanics. Goldstein[1] and Landau and Lifshitz[2] presume integrability implicitly without defining it. Arnol'd's[3] inspiring and informative book on classical mechanics discusses some ideas of deterministic chaos at a level aimed at advanced readers rather than at beginners, is short on driven dissipative systems, and his treatment of Hamiltonian systems relies on Cartan's formalism of exterior differential forms, requiring a degree of mathematical preparation that is neither typical nor necessary for physics and engineering graduate students.

The old Lie–Jacobi idea of complete integrability ('integrability') is the reduction of the solution of a dynamical system to a complete set of independent integrations via a coordinate transformation ('reduction to quadratures'). The related organizing principle, invariance and symmetry, is also expressed by using coordinate transformations. Coordinate transformations and geometry therefore determine the method of this text. For the mathematically inclined reader, the language of this text is not 'coordinate-free', but the method is effectively coordinate-free and can be classified as qualitative methods in classical mechanics combined with a Lie–Jacobi approach to integrability. We use coordinates explicitly in the physics tradition rather than straining to achieve the most abstract (and thereby also most unreadable and least useful) presentation possible. The discussion of integrability takes us along a smooth path from the Kepler problem, Galilean trajectories, the isotropic harmonic oscillator, and rigid bodies to attractors, chaos, strange attractors, and period doubling, and also to Einstein's geometric theory of gravity in space-time.

Because we emphasize qualitative methods and their application to the interpretation of numerical integrations, we introduce and use Poincaré's qualitative method of analysis of flows in phase space, and also the most elementary ideas from Lie's theory of continuous transformations. For the explicit evaluation of known integrals, the reader can consult Whittaker[4], who has catalogued nearly all of the exactly solvable integrable problems of both particle mechanics and rigid body theory. Other problems that are integrable in terms of known, tabulated functions are discussed in Arnol'd's *Dynamical Systems III*.[5]

The standard method of complete solution via conservation laws is taught in standard mechanics texts and implicitly presumes integrability without ever stating that. This method is no longer taught in mathematics courses that emphasize linear differential equations. The traditional teaching of mechanics still reflects (weakly, because very, very incompletely) the understanding of nonlinear differential equations before the advent of quantum theory and the consequent concentration on second order linear equations. Because the older method solves integrable *nonlinear* equations, it provides a superior

starting point for modern nonlinear dynamics than does the typical course in linear differential equations or 'mathematical methods in physics'. Most mechanics texts don't make the reader aware of this because they don't offer an adequate (or any) discussion of integrability vs nonintegrability. We include dissipative as well as conservative systems, and we also discuss conservative systems that are not Hamiltonian.

As Lie and Jacobi knew, planar dynamical systems, including dissipative ones, have a conservation law. We deviate from Arnol'd by relaxing the arbitrary requirement of analyticity in defining conservation laws. Conservation laws for dissipative systems are typically multivalued functions. We also explain why phase space is a *Cartesian* space with a scalar product rule (inner product space), but without any idea of the distance between two points (no metric is assumed unless the phase space actually is Euclidean). By 'Cartesian space', therefore, we mean simply a Euclidean space with or without a metric. Finally, the development of the geometric ideas of affine connections and curvature in chapter 11 allows us to use global integrability of the simplest possible flow to define phase space as a flat manifold in chapter 13. The language of general relativity and the language of dynamical systems theory have much geometric overlap: in both cases, one eventually studies the motion of a point particle on a manifold that is generally curved.

After the quantum revolution in 1925, the physics community put classical mechanics on the back burner and turned off the gas. Following Heisenberg's initial revolutionary discovery, Born and Jordan, and also Dirac discovered the quantum commutation rules and seemingly saved physicists from having to worry further about the insurmountable difficulties presented by the unsolved Newtonian three-body problem and other nonintegrable nonlinear dynamics problems (like the double pendulum). Einstein[6] had pointed out in 1917 that the Bohr–Sommerfeld quantization rules are restricted to systems that have periodic trajectories that lie on tori in phase space, so that quantization rules other than Bohr–Sommerfeld are required for systems like the three-body problem and mixing systems that yield Gibbs' microcanonical ensemble. The interesting difficulties posed by nonintegrability (or incomplete integrability) in both classical and quantum mechanics were conveniently forgotten or completely ignored by all but a few physicists from 1925 until the mid-1970s, in spite of a 1933 paper by Koopman and von Neumann where the 'mixing' class of two degree of freedom Hamiltonian systems is shown not to admit action-angle variables because of an 'everywhere dense chaos'.

This is the machine age. One can use the computer to evaluate integrals and also to study dynamical systems. However, while it is easy to program a computer and generate wrong answers for dynamics problems (see section 6.3 for an integrable example), only a researcher with a strong *qualitative* understanding of mechanics and differential equations is likely to be prepared to compute numbers that are both correct and meaningful. An example of a correct calculation is a computation, with controlled decimal precision, of unstable periodic orbits of a chaotic system. Unstable periodic orbits are useful for understanding both classical and quantum dynamics. An example of a wrong computation is a long orbit calculation that is carried out in floating-point arithmetic. Incorrect results of this kind are so common and so widely advertised in both advanced and popular books and articles on chaos theory that the presentation of bad arithmetic as if it were science has even been criticized in the media.[7] It is a disease of the early stage of the information age that many people labor under the comfortable illusion that bad computer arithmetic is suddenly and magically justified whenever a

system is chaotic.[8] That would be remarkable, if true, but of course it isn't.

The mathematics level of the text assumes that a well-prepared reader will have had a course in linear algebra, and one in differential equations that has covered power series and Picard's iterative method. The necessary linear algebra is developed in chapters 1 and 8. It is assumed that the reader knows nothing more about nonlinear systems of differential equations: the plotting of phase portraits and even the analysis of linear systems are developed systematically in chapter 3. If the reader has no familiarity with complex analysis then one idea must be accepted rather than understood: radii of convergence of series solutions of phase flows are generally due to singularities on the imaginary time axis.

Chapter 1 begins with Gibbs' notation for vectors and then introduces the notation of linear algebra near the end of the chapter. In chapter 2 and beyond we systematically formulate and study classical mechanics by using the same language and notation that are normally used in relativity theory and quantum mechanics, the language of linear algebra and tensors. This is more convenient for performing coordinate transformations. An added benefit is that the reader can more clearly distinguish superficial from nonsuperficial differences with these two generalizations of classical mechanics. In some later chapters Gibbs' notation is also used, in part, because of convenience.

Several chapter sequences are possible, as not every chapter is prerequisite for the one that follows. At the University of Houston, mechanics is taught as a one-year course but the following one-semester-course chapter sequences are also possible (where some sections of several chapters are not covered): 1, 2, 3, 4, 5, 7, 8, 9, 15 (a more traditional sequence): 1, 2, 3, 4, 6, 8, 9, 12, 13 (integrability and chaos), 1, 2, 3, 4, 6, 7, 12, 15, 16, 17 (integrable and nonintegrable canonical systems): 1, 2, 3, 4, 8, 9, 12, 13, 14 (integrability and chaos in noncanonical systems), or 1, 2, 3, 4, 5, or 6, 8, 9, 10, 11 (integrability and relativity).

The historic approach to mechanics in the first chapter was motivated by different comments and observations. The first was Herbert Wagner's informal remark that beginning German students are well prepared in linear algebra and differential equations, but often have difficulty with the idea of force and also are not familiar with the definition of a vector as it is presented in elementary physics. Margaret and Hugh Miller made me aware that there are philosophers, mathematicians, and others who continue to take Aristotle seriously. Postmodernists and deconstructionists teach that there are no universal laws, and the interpretation of a text is rather arbitrary. The marriage of cosmology and particle physics has led to the return to Platonism within that science, because of predictions that likely can never be decided empirically. Systems theorists and economists implicitly assume that commodity prices[9] and the movement of money define dynamical variables that obey mathematically precise time-evolution laws in the same sense that a collection of inanimate gravitating bodies or colliding billiard balls define dynamical variables that obey Newton's universal laws of motion. Other system theorists assert that 'laws of nature' should somehow 'emerge' from the behavior of 'complex systems'. Awareness of Plato's anti-empiricism and Aristotle's wrong ideas about motion leads to greater consciousness of the fact that the law of inertia, and consequently physics, is firmly grounded in empirically discovered universal laws of kinematics that reflect geometric invariance principles.[10] Beliefs about 'invisible hands', 'efficient markets', and 'rational expectations with infinite foresight'[11] are sometimes taught mathematically as if they might *also* represent laws of nature, in spite of the lack of any empirical evidence that the economy and other

behavioral phenomena obey any known set of mathematical formulae beyond the stage of the crudest level of curve-fitting. Understanding the distinction between the weather, a complex Newtonian system subject to universal law, and mere modelling based upon the statistics of irregularities that arise from effectively (mathematically) lawless behavior like stock trading might help to free economists, and perhaps other behaviorists, from the illusion that it is possible either to predict or even to understand the future of humanity in advance from a purely mathematical standpoint, and would also teach postmodernists the difference between physics, guesswork-modelling and simulations, and pseudo-science.

I am very grateful to Harry Thomas for many stimulating, critical and therefore very helpful discussions on geometry and integrability, and for sharing with me his lecture notes on a part of the proof of Bertrand's theorem. I'm also grateful to his colleague and former student Gerhard Müller for making me aware that the double pendulum provides a relatively elementary example of a nonintegrable and chaotic Newtonian system. I am extremely grateful to Eigil Anderson for introducing me to the description of Lie's work in E. T. Bell's history of mathematics in 1992: without having read the simple sentence by Bell stating that Lie showed that every one-parameter transformation on two variables has an invariant, the main organizing principle of this book (integrability) would have remained too primitive to be effective. I am grateful to my old friend and colleague, Julian Palmore, for pointing me to his paper with Scott Burns where the conservation law for the damped linear harmonic oscillator is derived. Julian also read and criticized my heuristic definitions of solvability and global integrability and helped me to make them sharper than they were before, but they are certainly not as precise as he would have liked them to be. I have also followed his advice, given several years ago, to emphasize flows in phase space. Dyt Schiller at Siegen, Finn Ravndal in Oslo, and Herbert Wagner in München have provided me with useful general criticism of earlier versions of the manuscript. George Reiter read sections 1.1, 1.2, and 18.1 critically. I am very grateful to copy editor Jo Clegg for a careful reading of the original manuscript, to Sigmund Stintzing for proofreading chapters 9 and 15 of the typescript, and especially to Herbert Wagner and Ludwig Maxmillians Universität (München) for guestfriendship during the fall 1996 semester of my sabbatical, when the typescript was proofed and corrected. I have enjoyed a lecture by Per Bak on long-range predictability vs qualitative understanding in physics, have also benefited from conversations with Mike Fields and Marty Golubitsky on flows and conservation laws, and from a discussion of differential forms with Bob Kiehn. I am grateful to Gemunu Gunaratne for many discussions and critical questions, to J. M. Wilkes for a 1977 letter on rotating frames, to Robert Helleman for pointing me to Arnol'd's discussion of normal forms. I am especially grateful to many of the physics and engineering students at the University of Houston who have taken my classical mechanics and nonlinear dynamics courses over the last several years. Chapter 8 was read and corrected, in part, by graduate students Rebecca Ward Forrest and Qiang Gao. The computer graphics were produced as homework exercises by Ola Markussen, Rebecca Forrest, Youren Chen, and Dwight Ritums. I would also like to thank Linda Joki for very friendly help with some of the typing. Special thanks go to Simon Capelin, Publishing Director for the Physical Sciences at Cambridge University Press, for his support and encouragement.

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