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SOME PROBLEMS ON THREE-DIMENSIONAL GRADED DOMAINS

M. Artin

1. Introduction.

One of the important motivating problems for ring theory is to describe the rings which have some of the properties of commutative rings. In this talk we consider this problem for graded domains of dimension 3. The conjectures we present are based on ideas of my friends, especially of Toby Stafford, Michel Van den Bergh, and James Zhang. However, they may not be willing to risk making them, because only fragments of a theory exist at present. Everything here should be taken with a grain of salt. I am especially indebted to Toby Stafford for showing me some rings constructed from differential operators which I had overlooked in earlier versions of this manuscript.

To simplify our statements, we assume throughout that the ground field k is algebraically closed and of characteristic zero, and that our graded domain A is generated by finitely many elements of degree 1. The properties which we single out are:

1.1.
(i) A is noetherian,
(ii) there is a dualizing complex ω for A such that the Auslander conditions hold, and
(iii) the Gelfand-Kirillov dimension behaves as predicted by commutative algebra.

A *dualizing complex* ω is a complex of bimodules such that the functor $M \mapsto M^D = \underline{\mathrm{RHom}}(M, \omega)$ defines a duality between the derived categories of bounded complexes of finite left and right A-modules. We will require it to be *balanced* in the sense of Yekutieli, which means that k^D is the appropiate shift of k (see [Aj,Y1,Y2] for the precise definitions). A graded ring A with a dualizing complex satisfies the *Auslander conditions* if for any finite A-module M and any submodule $N \subset \mathrm{Ext}^q(M, \omega)$, $\mathrm{Ext}^p(N, \omega) = 0$ for $p < q$ [Bj,Le,Y1,Y2,ASZ].

Let

$$j(M) = min\{j|\operatorname{Ext}^j(M,\omega) \neq 0)\}.$$

For domains of dimension 3, the link with Gelfand-Kirillov dimension is that $\mathrm{gk}(M) = 3 - j(M)$ [Le,Ye2,YZ]. Actually, the GK-dimension is not always the right dimension to use (see [ASZ]), but it will suffice for our purposes.

Though the properties listed above are central, they will appear only implicitly in what follows. All of them hold when A is commutative. So our problem becomes: Which graded domains satisfy these conditions? An answer to this question might take the form of an axiomatic description, or of a classification. This talk concerns classification, for which I should apologize. Maurice looked askance at what he might have called "botany", so the topic is not very suitable for the Auslander Conference.

Let's begin by reviewing the commutative case. If a commutative graded algebra A is written as a quotient $k[x_0, ..., x_n]/I$, where I is an ideal generated by some homogeneous polynomials $f_1, ..., f_r$, then its associated projective scheme $X = \operatorname{Proj} A$ is the locus of common zeros of $f_1, ..., f_r$ in projective space $\mathbb{P}^n$. Conversely, if we are given a projective scheme X, we can recover a graded algebra A as follows: For $n \gg 0$, A_n is the space of all functions on X with pole of order $\leq n$ at infinity. (The relations between A and $\operatorname{Proj} A$ hold only in large degree.)

In order to proceed, we need to rewrite this description in terms of sections of invertible sheaves. Let L denote the invertible sheaf of locally defined functions on X with pole of order ≤ 1 at infinity. Then $L^{\otimes n}$ is the sheaf of local functions with pole of order $\leq n$, so we can identify global functions with pole of order $\leq n$ at infinity with global sections of this sheaf: For $n \gg 0$, $A_n = H^0(X, L^{\otimes n})$. Multiplication in A is induced by the tensor product on L.

Van den Bergh [AV] has shown how to extend this description to construct noncommutative rings. He observes that in order for $L^{\otimes n}$ to be defined, L must have both a left and a right module structure over the structure sheaf $\mathcal{O}_X$, i.e., it must be an $(\mathcal{O}, \mathcal{O})$-bimodule. It is not necessary that the actions on the left and on the right agree; in fact this would be inconsistent if $\mathcal{O}$ weren't commutative. But if L is a bimodule, invertible as left and as right module, then $L^{\otimes n}$ is defined, and setting $B_n = H^0(X, L^{\otimes n})$ yields a graded algebra B which is often noncommutative. Of course, in order that B have reasonable properties, the bimodule L must satisfy a condition analogous to "ampleness" of an invertible sheaf in the commutative case (see [AV]).

The use of bimodules to define a polarization extends to certain noncommutative schemes X as well. But when X is commutative, it is not difficult to see that the left and right actions on an invertible bimodule L differ by an automorphism τ of the scheme X. In other words, the right action will be obtained from the left one by the rule

$$vf = f^{\tau}v,$$

for $v \in L$ and $f \in \mathcal{O}$. The bimodule obtained in this way from an automorphism will be denoted by L_τ, and we will use the notation $B(X, \tau, L)$ for the algebra defined in this way, i.e., the algebra whose part of degree n is $B_n = H^0(X, L_\tau^{\otimes n})$.

We now consider a noncommutative graded domain A. Thanks to the work of Stafford [ASt], the case that A has GK-dimension 2 is well understood:

Theorem 1.2. *Let B be a graded domain of GK-dimension 2 which is finitely generated by elements of degree 1. Then:*

(i) $\operatorname{Proj} B$ *is a commutative algebraic curve. More precisely, there is a projective algebraic curve C, an automorphism τ of C, and an invertible sheaf L of positive degree on C such that, for large n,* $B_n \cong H^0(C, L_\tau^{\otimes n})$.

(ii) The algebra B has the desired properties 1.1.

Note that this theorem is free of extraneous hypotheses, except for the requirement that A be generated in degree 1, which may seem artificial. In fact, as is explained in [ASt], the situation becomes considerably more complicated when this requirement is dropped.

Following the example of the Italian algebraic geometers at the end of the last century, we may attempt to classify the noncommutative projective surfaces which arise as $\operatorname{Proj} A$, for certain graded domains of GK-dimension 3. The object of this talk is to present a conjecture about them.

2. Examples of graded domains of GK-dimension 3.

There are many examples which show that Theorem 1.2 does not extend directly to higher dimension. It is true that one can construct noncommutative domains of GK-dimension 3 analogous to those of GK-dimension 2 by means of a suitable commutative algebraic surface X, automorphism τ, and invertible sheaf L. But other noncommutative domains exist, and those are the ones that one would like to describe. Here are four basic types:

Example 2.1. *Algebras finite over their centers.*

Algebras which are finite over their centers can be constructed quite simply from orders. Let K be the function field of a projective algebraic surface

S, and let D be a division ring with center K and finite over K. Let $\mathcal{A}$ be an $\mathcal{O}_S$-order in D. One obtains a noncommutative scheme $X = \underline{\mathrm{Spec}}\,\mathcal{A}$ by gluing the rings of sections of $\mathcal{A}$ over affine open sets of Z, using central localizations. Then a graded ring A finite over its center can be defined as follows: Let L denote the sheaf $\mathcal{O}(1)$ on S, and set $\mathcal{A}(n) = \mathcal{A} \otimes_{\mathcal{O}} L^{\otimes n} \approx \mathcal{A}(1) \otimes_{\mathcal{A}} \otimes \cdots \otimes_{\mathcal{A}} \mathcal{A}(1)$. Then $A_n = H^0(S, \mathcal{A}(n))$. Other rings, not necessarily finite over their centers, can be constructed using $(\mathcal{A}, \mathcal{A})$-bimodules which which are not central.

Example 2.2. *Auslander regular algebras of dimension* 3.

A graded domain A of finite global dimension which satisfies the Auslander conditions is called an Auslander regular algebra (see[Le]). The Auslander regular algebras of dimension 3 have been classified completely [ASch,ATV], and Schelter's *Sklyanin algebras* are the most interesting ones among them. These are the three-dimensional analogues of some well-known four-dimensional algebras defined by Sklyanin [Skl]. A Sklyanin algebra $A = A(E, \sigma)$ of dimension 3 can be defined in a somewhat mysterious way, in terms of an elliptic curve E embedded as a cubic in $\mathbb{P}^2$ and a translation σ of E. The associated projective scheme $\mathrm{Proj}\, A$ is a deformation of the projective plane: a *quantum plane* [Ar]. The other Auslander regular algebras of dimension 3, and their associated quantum planes, are obtained from automorphisms of singular plane cubics.

Example 2.3. *Polynomial extensions of domains of GK-dimension* 2.

Here B is a graded domain of GK-dimension 2 and $A = B[z]$, where z is a central variable of degree 1. By Theorem 1.2, B has the form of a twisted homogeneous coordinate ring of a curve: $B = B(C, \tau, L)$. If the automorphism τ of C has infinite order, then neither B nor A is PI. In this case, C will be rational or elliptic, and if it is elliptic, then τ will be a translation. The case that τ is a translation of infinite order on an elliptic curve is especially interesting.

Example 2.4. *Homogenized differential operator rings.*

This example is due to Stafford. Let C be a nonsingular curve with structure sheaf $\mathcal{O}$, and let $\mathcal{D}$ denote the sheaf of rings of differential operators on C. If x denotes a local parameter at a point $p \in C$, then locally at p, $\mathcal{D}$ has the form $\mathcal{O}_p\langle y \rangle$, where y is the derivation $\frac{d}{dx}$, and $yx = xy + 1$. We choose a point p_0 on C, and consider the subsheaf $\mathcal{D}'$ of $\mathcal{D}$ which is equal to $\mathcal{D}$ except at the point p_0, and which is generated by $y_0 = x_0 \frac{d}{dx_0}$ at that point, x_0 being a local parameter. Thus the relation $y_0 x_0 = x_0 y_0 + x_0$ holds at p_0. This relation shows that, locally, x_0 is a normal element of $\mathcal{D}'$, which is the reason that we replace $\mathcal{D}$ by $\mathcal{D}'$. We homogenize the defining relation of $\mathcal{D}'$ using a central variable z, to obtain a sheaf of graded rings $\mathcal{A}$ which has the local form $\mathcal{O}_p\langle y,z\rangle$, with the defining relation $y_0x_0 = x_0y_0 + x_0z$ at p_0 and $yx = xy + z$ at all other points. The subsheaf $\mathcal{A}_n$ of elements of degree n is isomorphic to the sheaf of differential operators which are in $\mathcal{D}'$ and which have order $\leq n$. Locally at the point p_0, the element x_0 normalizes $\mathcal{A}$. We choose an integer $r > 2g-2$ and we denote by $\mathcal{L}_n$ the right $\mathcal{O}$-module of sections of $\mathcal{A}_n$ with pole of order nr at p_0. So locally at p_0, a section of $\mathcal{L}_n$ can be written in the form $\alpha_n x_0^{-nr}$, where $\alpha_n \in \mathcal{A}_n$. Because x_0 is normalizing, multiplication sends $\mathcal{L}_m \times \mathcal{L}_n \to \mathcal{L}_{m+n}$, so we obtain a graded ring A by setting $A_n = H^0(C, \mathcal{L}_n)$.

There are many other interesting graded domains of dimension 3, for example the *quantum quadrics* which are obtained from Sklyanin algebras of dimension 4 by dividing by a central element of degree 2 [Sm, SStd].

3. A General Description of $X = \operatorname{Proj} A$.

Let A be a noetherian graded ring, and let $\mathcal{C}$ denote the category of finitely generated, graded, right A modules, modulo the subcategory of torsion modules (modules finite-dimensional over k). This category $\mathcal{C}$ = gr-$A/(torsion)$ can also be described as the category of *tails* $M_{\gg 0}$ of finitely generated graded A-modules. By definition, the projective scheme $X = \operatorname{Proj} A$ associated to A is the triple $(\mathcal{C}, \mathcal{O}, s)$, where $\mathcal{O}$ is image in $\mathcal{C}$ of the right module A_A, and s is the autoequivalence of $\mathcal{C}$ defined by the shift operator on graded modules [AZ,Ma,Ve]. Working out the consequences of this definition is an ongoing program, and we will not need to consider it in detail. However, we need to review some geometric concepts, namely of points and fat points. Following tradition, we assume that $X = \operatorname{Proj} A$ is smooth. This means that $\mathcal{C}$ has finite injective dimension, or that for every finite graded A-module M and for $q > 2$, the graded injectives I^q in a minimal resolution $0 \to M \to I^0 \to I^1 \to \cdots$ are sums of the injective hull of k.

Our conditions 1.1 imply that the ring A can be recovered, in sufficiently large degree, from its associated projective scheme. This means that Zhang's condition χ, that $\operatorname{Ext}^q(k, M)$ is finite dimensional for all finite modules M, holds for A [YZ].

The tail of a critical module M of GK-dimension 1 is called a *fat point* of X. Fat points are the projective analogs of finite dimensional representations of a ring [Sm,SSts]. If the stable Hilbert function $\dim M_n$, $n \gg 0$, is the constant function 1, so that M has multiplicity 1, then the tail $M_{\gg 0}$ is called a *point* of X.

We say that X is finite over its center if there is a commutative algebraic surface S and a coherent $\mathcal{O}_S$-algebra $\mathcal{A}$ such that X is isomorphic to the relative scheme $\underline{\mathrm{Spec}}\,\mathcal{A}$ over S, as in Example 2.1. Specifically, this means that $\mathcal{C}$ is equivalent to the category of coherent as $\mathcal{O}_S$-modules with a right $\mathcal{A}$-module structure, and that s is an ample autoequivalence of $\mathcal{C}$.

In all known examples of graded domains of GK-dimension 3 which satisfy 1.1 and such that $X = \mathrm{Proj}\,A$ is *not* finite over its center, the scheme X has some remarkable properties:

3.1. There is a quotient $B = A/I$ of A of pure GK-dimension 2, such that:

(i) $Y = \mathrm{Proj}\,B$ is a commutative projective curve, possibly reducible, whose points are points of X.

(ii) X has only finitely many fat points in addition to the points on Y. In particular, X has only finitely many fat points of multiplicity $\neq 1$.

(iii) The complement of Y in X is an affine open subscheme. In other words, there is a finitely generated noetherian domain R such that $X - Y = \mathrm{Spec}\,R$, or that the category mod-R of finite R-modules is equivalent with the quotient category gr-$A/(I -$ torsion).

A priori, it is not clear that $\mathrm{Proj}\,A$ should have any points at all. This is a puzzling point.

4. A Conjecture.

The graded quotient ring of a graded Ore domain A has the form $Q(A) = D[z, z^{-1}, \phi]$, where D is a division ring and ϕ is an automorphism of D [NV]. We will refer to the division ring D as the *function field* of the scheme $X = \mathrm{Proj}\,A$, and we will say that two such schemes X, X' are *birationally equivalent* if their function fields are isomorphic extensions of k. If X is birational to a quantum plane (Example 2.2) we call it a *q-rational* surface. If X is birational to one of the surfaces listed in Examples 2.3,2.4 and in which the curve C has genus $g > 0$, we call it *q-ruled*. (There is an intrinsic definition of q-ruled surface in terms of "bimodule algebras" over a curve [VdB1].)

Conjecture 4.1. *Let k be an algebraically closed field of characteristic zero, and let A be k-algebra of GK-dimension 3 satisfying the properties 1.1. Then $X = \mathrm{Proj}\,A$ is birationally equivalent to $\mathrm{Proj}\,A'$, where A' is one of the graded domains described in Examples 2.1-2.4. So one of the following holds:*

(i) X is finite over its center,

(ii) X is q-rational, or

(iii) X is q-ruled.

The properties 1.1 are included as hypotheses in this conjecture. Ideally, we would like them to be consequences of more basic assumptions on the structure of A, as is the case in dimension 2 (see Theorem 1.2). However, we don't know what the necessary assumptions are. At this stage of our knowledge, any reasonable hypotheses on the structure are acceptable.

A finer classification would subdivide (i) into two classes:

4.2.
(ia) A finite over its center, and
(ib) A not finite over its center, but X finite over its center.

The possibilities for (ib) can be enumerated conjecturally as well. They include cases in which X is a commutative surface, such as an abelian surface, which has a continuous group of automorphisms. (See the last section of [AV] in this connection.)

It is interesting to note that PI algebras appear as a natural class of rings in Conjecture 4.1. Indeed, the PI case (ia) should be viewed as the "general" one. It corresponds roughly to the class of commutative surfaces of Kodaira dimension ≥ 0, though PI algebras also appear as special cases in (ii),(iii). There is no hope of listing these rings.

For those of us who are interested in describing noncommutative phenomena, it may, at first glance, seem a bit disappointing to think that rational and ruled surfaces could be the only ones which have noncommutative analogues not finite over their centers. One must remember that the most beautiful results of the Italian school, such as the numerical characterizations of rational and ruled surfaces of Castelnuovo and Enriques, and Castelnuovo's theorem on the rationality of plane involutions, concern precisely these surfaces. Whether the conjecture is correct or not, extending those results to the noncommutative setting is a worthy goal for people working in ring theory.

5. The Division Rings.

Since Conjecture 4.1 concerns only the birational equivalence classes of noncommutative surfaces, it can be stated in terms of their function fields. Here is a list of the division rings which are predicted by the conjecture:

List of Division Rings 5.1. *In this list, k is assumed algebraically closed, of characteristic zero, σ denotes a translation by a point of infinite order of an elliptic curve E, and $q \in k^*$ is not a root of unity.*

1. *division rings which are finite algebras over function fields of transcendence degree 2.*

2. *q-rational division rings:*
(a) *$k_q(x,y)$, the field of fractions of the q-plane $yx = qxy$.*

(b) *the Sklyanin division ring $S(E,\sigma)$, the degree zero part of the graded field of fractions of the Sklyanin algebra $A(E,\sigma)$.*
(c) *D_1, the field of fractions of the Weyl algebra.*

3. *q-ruled division rings:*
(a) *$K(E,\sigma)$, the field of fractions of the Ore extension $k(E)[t,\sigma]$.*
(b) *$D(C)$, the field of fractions of the ring of differential operators on a curve C of positive genus.*

It is interesting to note that Schelter's 3-dimensional Sklyanin algebras provide the only division rings on our list which are relatively new. In fact, Van den Bergh showed recently that $S(E,\sigma)$ is the ring of invariants in $K(E,\sigma)$ under the involution defined by the map which sends $p \mapsto -p$ in the group E, and $t \mapsto t^{-1}$. Thus the Sklyanin division rings are also closely related to more classical ones.

There are several definitions of dimension for division rings. The first one, the GK transcendence degree, was introduced by Gelfand and Kirillov [GK,Z1]. They used this notion to distinguish the fields of fractions of the Weyl algebras A_n. One can also define the dimension of D to be its projective dimension as a module over $D \otimes D^{opp}$ [Re,Ro,St]. Recently Zhang [Z2] found an elegant definition (ZD) for which it is easier to prove some general properties: Let D be a division ring over a field k. Then $\mathrm{zd}(D) \geq r$ if there exists a finite dimensional k-subspace V of D containing 1 and a constant c such that for every finite dimensional subspace W of D,

$$\dim(VW) \geq \dim(W) + c\dim(W)^{\frac{r-1}{r}}.$$

As Zhang points out, the way to understand this definition intuitively is to imagine $\dim(W)$ as the volume of a variable region in an r-dimensional space. Then with an appropriate constant factor, $\dim(W)^{\frac{r-1}{r}}$ is a lower bound for the volume of the boundary ∂W. The regions VW and W differ only near the boundary.

Zhang has shown that $\mathrm{zd}(D)$ is at most equal to the GK transcendence degree. It is not known whether or not the two are always equal. We will call a Z2 division ring one which is finitely generated over k and such that $\mathrm{zd}(D)$ is equal to 2.

Conjecture 5.2. *The list 5.1 contains all $Z2$ division rings.*

Zhang has shown that the division rings listed are distinct, and that certain inclusions among them can not occur. For example, $k_q(x,y)$ is not a subfield of D_1. (Some of these facts were known before.) A convenient tool for verifying them is the concept of *prime divisor*. A prime divisor of a Z2-division ring D is a discrete valuation ring R whose residue field is a function field in one variable over k. The components of the point locus Y of $X = \operatorname{Proj} A$ (see 3.1) define valuations of its function field, the ones classically referred to as being "of the first kind" on X.

Proposition 5.3. *Let D be one of the division rings 5.1, and suppose that D is not finite over its center. Then D has at least one prime divisor. More precisely,*

- *(i) The prime divisors of $k_q(x,y)$ are determined by the values $v(x)$,$v(y)$, which can be any pair of relatively prime integers. The residue field of every prime divisor is a rational function field.*
- *(ii) The Sklyanin division ring $S(E,\sigma)$ has exactly one prime divisor. Its residue field is the function field $k(E)$ of the elliptic curve.*
- *(iii) The residue field of every prime divisor of D_1 is a rational function field.*
- *(iv) The division ring $K(E,\sigma)$ has exactly two prime divisors. Both have the function field $k(E)$ as residue field.*
- *(v) There is exactly one prime divisor of $D(C)$ whose residue field is the function field $k(C)$. All other prime divisors have rational residue fields.*

A prime divisor R comes equipped with an outer automorphism, which is defined by conjugating by a generator of its maximal ideal M, and which provides further information about the division ring. It also has an *index*, the largest integer n such that R/M^n is commutative.

Assertion 5.3(i) is due to Zhang, and Willaert [W] has studied prime divisors in D_1. A proof of 5.3(ii) is outlined in Section 6. We don't know how to prove the existence of a prime divisor directly from the Z2 condition, and indeed, even if one always exists, it may be difficult to give a direct proof because the assertion is false when the assumption that D is not finite over its center is removed. A proof of the existence might be a starting point for classifying the Z2 division rings.

Some heuristic evidence for the conjecture that a Z2 division ring has a prime divisor is provided by the following construction, which will produce one in a few cases: Choose generators for a convenient finitely generated subring S of D, and form a graded ring A by homogenizing the defining relations, using a central variable z of degree 1. If we are very lucky, A will be noetherian and of GK-dimension 3, and z will generate a completely prime ideal P. Then Theorem 1.2 identifies A/P as a twisted homogeneous coordinate ring of a curve. In that case P will be localizable, and the graded localization A_P will be a graded valuation ring, whose subring of degree zero is the required prime divisor.

The set $\mathcal{P}_1$ of prime divisors of the Weyl skew field D_1 forms a fairly complicated picture, but one can give a combinatorial description in terms of the birational geometry of the ordinary projective plane $\mathbb{P}^2$ (see also [W]). Consider prime divisors of the rational function field $k(x, y)$ which are centered on the line L at infinity in $\mathbb{P}^2$. Define the index of such a prime divisor to be the order of pole of the double differential $dx\,dy$, and let $\mathcal{P}_2$ denote the set of prime divisors in $k(x, y)$ of positive index.

Proposition 5.4. *There is a bijective map $\mathcal{P}_1 \to \mathcal{P}_2$ which preserves index.*

A similar description can be given for the prime divisors of $D(C)$. This proposition was proved in joint work with Stafford. It is not very difficult, but is too long to include here.

6. Evidence.

Besides the examples, Conjecture 4.1 is based on evidence collected by three methods:

6.1.
(1) quantization, or deformation of commutative schemes,
(2) the theory of orders and the Brauer group, and
(3) Van den Bergh's notion of noncommutative blowing up.

We have no additional evidence on which the rash Conjecture 5.2 that our list of Z2 fields is complete can be based. It is just that no other division rings have appeared up to now.

Discussion of the evidence.

(1) It is reasonable to suppose that a sizable family of noncommutative surfaces would leave a trace as a "classical limit", a commutative scheme. If the conjecture is correct, then the limit surface must be rational or ruled. We may test this conclusion by studying infinitesimal deformations of a commutative surface. As is well known, the main invariant of a first order deformation of a commutative surface X_0 is its Poisson bracket, which is a section of the anticanonical bundle $\wedge^2 T_{X_0} = \mathcal{O}_{X_0}(-K)$. On many surfaces, this bundle has no sections. The first assertion of the following proposition follows directly from the classification of commutative surfaces (see [Be,BPV]). A proof of the second assertion is outlined in Section 8.

Proposition 6.2. *Let X_0 be a smooth projective surface which admits a noncommutative infinitesimal deformation X. Then*
(i) X_0 has an effective anticanonical divisor, and is one of the following: a rational surface, a birationally ruled surface, an abelian surface, or a K3 surface.

(ii) *If there exists an ample invertible sheaf on X_0 which extends to an invertible bimodule on X, then X_0 is rational or birationally ruled.*

The existence of an ample invertible bimodule is necessary in order for the polarization of X_0 to extend to the deformation, i.e., for the homogeneous coordinate ring to deform compatibly (see 8.2 for a precise statement). Thus the classical limit surfaces are of the expected types.

As is well known, the anticanonical divisors on a surface have arithmetic genus 1. Those on ruled surfaces are described by the next proposition.

Proposition 6.3. *Let Z be an effective anticanonical divisor on a ruled surface X over a curve C of genus g.*

(i) *If $g > 1$, $Z = 2D + F$, where D is a section and $F = \sum F_i$ is a sum of rulings.*

(ii) *If $g = 1$, then either Z has the above form, or else Z is the sum of two disjoint sections.*

Examples 2.3 and 2.4 are deformations of commutative surfaces determined by Poisson brackets of the forms (ii) and (i) respectively.

(2) Studying orders can provide heuristic evidence for the conjecture that various q-rational surfaces which arise, for example by quantization, should be birationally equivalent. To obtained this evidence, we specialize q to a root of unity or σ to a translation of finite order. Then, in all cases which have been investigated, the algebra A becomes finite over its center, and one can test birational equivalence using known results about the Brauer group. The description of deformations of orders is still being worked out, but Ingalls has shown that if a maximal order whose center is a smooth surface Z admits a non-PI deformation, then the anticanonical sheaf $\mathcal{O}_Z(-K)$ must have a nonzero section which vanishes on the ramification locus of the order. So the center Z of $X = \operatorname{Proj} A$ is one of the surfaces listed in 6.2(i). For instance, Z may be a rational surface with effective anti-canonical divisor, and the anticanonical divisor may be an elliptic curve. The next proposition is rather easy to prove:

Proposition 6.4. *A smooth elliptic curve E has an essentially unique embedding as a cubic curve in $\mathbb{P}^2$. Suppose that E is also embedded as an anticanonical divisor E_1 into a rational surface X_1. Then the pair $E_1 \subset X_1$ is birationally equivalent to the embedding $E_2 \subset X_2$ of E as a plane cubic in $X_2 = \mathbb{P}^2$. In other words, the local rings of X_i at the general points of E_i are isomorphic.*

A similar result holds when the anticanonical divisor E is a cycle of rational curves [Lo]. Now if E is an elliptic curve on a rational surface X and if E'/E is an etale covering of elliptic curves, then Brauer group computations [AM] show that there is a division ring D with center the rational function field $k(X)$, whose branching data is this given covering, and that D is unique up to $k(X)$-isomorphism. The proposition shows that, provided that E is anticanonical, D is isomorphic to the division ring obtained from the cubic embedding $E \subset \mathbb{P}^2$. This is what the conjecture would predict for the Sklyanin division ring, if σ were allowed to have finite order.

(3) Having plausibility arguments for the existence of birational maps between certain of the projective schemes, a natural question is: What are these birational maps? In the commutative case, a theorem of Zariski asserts that one can factor any birational transformation between smooth surfaces into a succession of blowings up and down. The key ingredient which has been provided by Van den Bergh [VdB2] is to describe the noncommutative analogue of the blowing up of a surface. He has shown that in favorable cases one can blow up a point p of the point locus of X, obtaining another projective scheme X' in which the point p is replaced by an exceptional module. He has also shown how the blowing up process produces the mysterious sporadic fat points which appear on special quantum quadrics (see [S],[SSts]).

Because blowing up is an essentially projective construction, the definition is subtle and the blowing up does not lead to a projective scheme in all cases. We refer to Van den Bergh's paper for the definitions. For the purposes of this paper, it seems sufficient to illustrate the process by an example. This is done in Section 9.

7. Prime Divisors of the Sklyanin Division Ring:

This section gives a proof of Proposition 5.3 (ii). We refer to the literature for known results about the 3-dimensional Sklyanin algebra $A(E,\sigma)$. Recall that σ is assumed of infinite order. Let A denote the 3-Veronese of $A(E,\sigma)$. There is a central element g of $A(E,\sigma)$ of degree 3 [ATV], and it has degree 1 in A. Let $Q = D[z, z^{-1}; \tau]$ denote the graded fraction field of A. One can take for z any element of A_1 (or of Q_1 for that matter). A change of the element $z \in Q_1$ changes τ by an inner automorphism. Since g is central, $\tau = 1$ when $z = g$. Thus τ is inner for all choices of z.

Let R be a prime divisor of D, a discrete valuation whose residue field $K = R/M$ is a function field in one variable, and let ν denote the associated valuation. A discrete valuation is stable under inner automorphism. Thus R is τ-stable, and $R[z, z^{-1}; \tau]$ is defined and is a subring of Q.

Lemma 7.1. *We may choose $z \in A_1$ so that $A_1 \subset Rz$. When this is done, $A \subset R[z; \tau]$.*

Proof. With g as above, we choose $u \in A_1 g^{-1}$ with $\nu(u)$ minimal. Then $A_1 g^{-1} u^{-1} \subset R$. So $z = ug$ has the required property.

We denote the automorphism of the residue field K of R which is induced by the action of τ on R by τ too, so that $K[z, z^{-1}; \tau] = R[z, z^{-1}; \tau] \otimes_R K$. Let $\overline{A}$ denote the image of A in $K[z, z^{-1}; \tau]$, and let π be the canonical homomorphism $A \to \overline{A}$. Thus $\overline{A}$ is a graded domain.

Lemma 7.2. $\overline{A} = A/gA$.

Proof. We use the fact that the Sklyanin algebra has no two-sided graded ideal $\overline{I}$ such that $\mathrm{gk}(\overline{A}/\overline{I}) = 1$. Since $K[z, z^{-1}; \tau]$ is a domain of GK-dimension 2, $\mathrm{gk}(\overline{A}) \leq 2$. It is at least 1 because $z \in A_1$, and it can't be 1. So $\mathrm{gk}(\overline{A}) = 2$, and $\overline{A}$ is the coordinate ring of a twisted curve, one of the rings described in Theorem 1.2. We also know that $\overline{A}$ has no graded ideal I such that $\mathrm{gk}(A/I) = 1$, because A has none. Therefore every nonzero ideal of $\overline{A}$ is cofinite. Let $\overline{g}$ be the residue in $\overline{A}$ of the central element g. If $\overline{g}$ were not 0, $\overline{A}/\overline{g}\overline{A}$ would have GK-dimension equal to $\mathrm{gk}(\overline{A}) - 1 = 1$. Since this is impossible, $\overline{g} = 0$ and g is in the kernel of π. Since g generates a completely prime ideal in A and $\mathrm{gk}(A/gA) = 2$, $\overline{A} = A/gA$.

The set S of homogeneous elements of A which are not divisible by g is an Ore set, and the degree zero part of the ring of fractions $S^{-1}A$ is the valuation ring of the *g-adic* valuation which, on A, is given by the rule $\nu(a) = r$ if $a = g^r b$ and g does not divide b. By what has been proved above, $S^{-1}A \subset R[z, z^{-1}; \tau]$, because the image of $s \in S$ in the graded division ring $K[z, z^{-1}; \tau]$ is not zero. Since $(S^{-1}A)_0$ is a valuation ring, this ring must be R.

8. Deformation of a Commutative Surface.

In order to keep the discussion brief, we restrict our attention to first order deformations, those parametrized by the ring $R = k[\epsilon]$, $\epsilon^2 = 0$. We denote by $\mathcal{A}_R$ the category of R-algebras A such that $A \otimes_R k$ is commutative.

Let us call a *scheme* X_R in $\mathcal{A}_R$ a commutative scheme X_0, together with an extension of its structure sheaf $\mathcal{O}_{X_0}$ to a sheaf of rings $\mathcal{O}_X$ in $\mathcal{A}_R$, compatibly with localization. The sheaf $\mathcal{O}_X$ will be called the *structure sheaf* of X. By *coherent sheaf* on an R-scheme X_R, we mean a sheaf of finite right $\mathcal{O}_X$-modules which is compatible with localization. The scheme X is *smooth* if $\mathcal{O}_X$ is flat over R and if X_0 is smooth. We write $\mathcal{O} = \mathcal{O}_X$, $\mathcal{O}_0 = \mathcal{O}_{X_0}$, and we denote the tangent sheaf on X_0 by T_0.

Because X_0 is commutative, the commutator $[x, y]$ on $\mathcal{O}$ can be viewed as a skew symmetric map $\alpha : \mathcal{O}_0 \times \mathcal{O}_0 \to \mathcal{O}_0$ which is a derivation in each variable. We call such a skew derivation a *bracket*. The next proposition is standard.

Proposition 8.1.

(i) *Let X_0 be a smooth scheme over k. The set of brackets on X_0 is classified by $H^0(X_0, \wedge^2 T_0)$.*

(ii) *If a bracket α is given, then a smooth extension of X_0 to R with commutator α exists locally. The obstruction to its existence globally lies in $H^2(X_0, T_0)$. If the obstruction vanishes, then the isomorphism classes of of extensions X whose commutators are the given bracket form a principal homogeneous space under $H^1(X_0, T_0)$.*

(iii) *For any smooth extension X, the sheaf $\underline{\mathrm{Aut}}(\mathcal{O})$ of local automorphisms of X which reduce to the identity on X_0 is isomorphic to T_0.*

The first assertion of Proposition 6.2 follows from (ii) and the classification of surfaces [Be,BPV].

Suppose that a smooth extension X is given, and that X_0 is projective, with ample line bundle L_0. We consider the problem of extending this polarization to X, so as to obtain a noncommutative projective scheme in the sense of [AZ]. What we want is an R-linear, ample autoequivalence s of the category mod-X of coherent sheaves over X which extends the polarization $s_0 = \cdot \otimes_{\mathcal{O}_0} L_0$ of X_0 defined by L_0. We call an $(\mathcal{O}, \mathcal{O})$-bimodule L *invertible* if R acts centrally on L, L is locally isomorphic to $\mathcal{O}$ as left and as right module, and $L_0 = L \otimes_R k$ is a central $\mathcal{O}_0$-bimodule.

Proposition 8.2. *Let X be a scheme in $\mathcal{A}_R$, and let L_0 be an invertible sheaf on X_0. Let s be an autoequivalence of mod-X which extends the autoequivalence s_0 of mod-X_0 defined by L_0.*

(i) *There is an invertible $\mathcal{O}$-bimodule L such that $s \cong \cdot \otimes_{\mathcal{O}} L$ and $L \otimes_R k \approx L_0$. If s_0 is ample, then so is s.*

(ii) *With L as above, set $A = \bigoplus H^0(X, L^{\otimes n})$ and $A^0 = \bigoplus H^0(X_0, L_0^{\otimes n})$. Then A is a noetherian graded R-algebra, $A_{\gg 0}$ is R-flat, and $A_{\gg 0} \otimes k \approx A^0_{\gg 0}$.*

We analyze the problem of extending $\mathcal{O}_0$-bimodule L_0 to $\mathcal{O}$ in two steps. First, we extend the right module structure. Right $\mathcal{O}$-modules locally isomorphic to $\mathcal{O}$ are classified by $H^1(X, \mathcal{O}^*)$, and there is an exact sequence

$$0 \to \mathcal{O}_0^+ \xrightarrow{1+\epsilon} \mathcal{O}^* \to \mathcal{O}_0^* \to 0.$$

Thus, as in the commutative case, the obstruction to extending the right module L_0 lies in $H^2(X, \mathcal{O}_0)$, and if it is zero, then the group $H^1(X, \mathcal{O}_0)$ operates transitively on the set of classes of extensions. This is a standard situation.

Next, we consider the left module structure of an invertible right module $L_{\mathcal{O}}$. The commutant $\mathcal{E} = \underline{\mathrm{End}} L_{\mathcal{O}}$ is locally isomorphic to $\mathcal{O}$. More precisely,