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Ergodic Theory of $\mathbb{Z}^d$ Actions

Proceedings of the Warwick Symposium 1993-4

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INTRODUCTION

These notes represent the proceedings of the Warwick Symposium on 'Ergodic Theory of $\mathbb{Z}^d$-actions' in 1993-94.

The classical theory of dynamical systems has tended to concentrate on $\mathbb{Z}$-actions or $\mathbb{R}$-actions (i.e. discrete or continuous 'time evolutions'). However, in recent years there has been considerable progress in the study of higher dimensional actions (i.e. actions of $\mathbb{Z}^d$ or $\mathbb{R}^d$ with $d > 1$). This progress was motivated not only by statistical physics, but also by the remarkable successes of multiple recurrence arguments in certain number theoretic problems, and by the intriguing rigidity properties of some classes of 'geometric' and 'algebraic' $\mathbb{Z}^d$-actions.

Historically, much of the interest in $\mathbb{Z}^d$-actions came from the study of classical lattice gas models (for example, the famous Ising model). In the simplest case where $d = 1$ (i.e. in the case of $\mathbb{Z}$-actions) this led to the development of the thermodynamic approach to the ergodic theory of Anosov and Axiom A systems during the 1960s and 1970s (cf. D. Ruelle's monograph *Thermodynamical Formalism*). By contrast, the corresponding problems of understanding the ergodic theory of even the simplest higher rank actions leads quickly to deep unsolved problems (for example, Furstenberg's conjecture on the invariant measures for $\times 2$ and $\times 3$ on the unit interval, or the undecidability problems associated with higher dimensional shifts of finite type). In the context of statistical mechanics, one of the manifestations of this difference between $d = 1$ and $d > 1$ is the existence of very complicated phase transitions in higher dimensions.

The exploitation of connections between multi-parameter ergodic theory and number theory originated largely in the now famous ergodic theory proof, due to Furstenburg, of Szemeredi's theorem on the existence of arithmetic progressions in sequences of integers of positive density. In 1978 Furstenburg showed how Szemeredi's theorem could be reduced to a problem of recurrence for measurable $\mathbb{Z}^d$-actions (cf. H. Furstenburg, *Recurrence in ergodic theory and combinatorial number theory*). This proof of Szemeredi's theorem initiated a highly successful programme which yielded important results both in the ergodic theory of $\mathbb{Z}^d$ actions and in number theory.

A third source of recent interest in $\mathbb{Z}^d$-actions is the study of certain geometric and algebraic examples of $\mathbb{Z}^d$-actions. The simplest actions of this kind are obtained by considering two or more commuting toral automorphisms. Such actions exhibit some remarkable rigidity properties (such as a very small centralizer of the action, or a scarcity of invariant measures and 'regular' cocycles), which were first encountered in the context of geometric actions of groups like $SL(d, \mathbb{Z})$. Recent work by Katok and others has shown that these rigidity properties are shared by a much wider class of $\mathbb{Z}^d$-actions by commuting diffeomorphisms of manifolds. The class of 'algebraic' $\mathbb{Z}^d$-actions mentioned above originated in a simple example proposed by Ledrappier and consists of all $\mathbb{Z}^d$-actions by commuting automorphisms of compact groups (cf. K. Schmidt, *Dynamical systems of algebraic origin*). This class allows the development of a quite detailed and coherent theory which goes far beyond what can be established for more general systems.

The surveys and original research articles in this volume cover many of these diverse ingredients of the theory of $\mathbb{Z}^d$-actions. The success of the symposium was due to the outstanding survey lecture series and individual research talks by many of the main contributors to these recent developments in multi-parameter ergodic theory, and to the expertise and enthusiasm of the staff of the Warwick Mathematics Research Centre (Peta McAllister and Elaine Coelho Greaves). On behalf of the other co-organizers (W. Parry and P. Walters) we would like to thank them all for their contributions.

During the symposium, Bill Parry celebrated his 60th birthday. As colleagues and friends we would like to dedicate these proceedings to him.

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