Ergodic Ramsey Theory–an Update

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0. Introduction.

This survey is an expanded version and elaboration of the material presented by the author at the *Workshop on Algebraic and Number Theoretic Aspects of Ergodic Theory* which was held in April 1994 as part of the 1993/1994 *Warwick Symposium on Dynamics of* $\mathbf{Z}^n$*-actions and their connections with Commutative Algebra, Number Theory and Statistical Mechanics.* The leitmotif of this paper is: Ramsey theory and ergodic theory of multiple recurrence are two beautiful, tightly intertwined and mutually perpetuating disciplines. The scope of the survey is mostly limited to Ramsey-theoretical and ergodic questions about $\mathbf{Z}^n$–partly because of the proclaimed goals of the Warwick Symposium and partly because of the author's hope that $\mathbf{Z}^n$-related combinatorics, number theory and ergodic theory can serve as an ideal lure through which the author's missionary zeal will reach as wide an audience of potential adherents to the subject as possible.

To compensate for the selective neglect of details and for the lack of full generality in some of the proofs, which were imposed by natural time and space limitations, a significant effort was spent on accentuation and motivation of ideas which lead to conjectures and techniques on which the proofs of conjectures hinge.

Here now is a brief description of the content of the five sections constituting the body of this survey. In Section 1 three main principles of Ramsey theory are introduced and their connection with the ergodic theory of multiple recurrence is emphasized. This section contains a lot of discussion and very few proofs. The goal in this section is to help create in the reader a feeling of what Ergodic Ramsey Theory is all about.

Section 2 is devoted to a multifaceted treatment of a special case of the polynomial ergodic Szemerédi theorem recently obtained in [BL1] (Theorem 1.19 of Section 1). Different approaches are discussed and brought to (hopefully) a convincing level of detail.

In Section 3 the somewhat esoteric, but fascinating and very useful object $\beta\mathbf{N}$, the Stone-Čech compactification of $\mathbf{N}$, is introduced and discussed in some detail. An ultrafilter proof of the celebrated Hindman's theorem is given and some applications of $\beta\mathbf{N}$ and Hindman's theorem to topological dynamics, especially to distal systems, are discussed. This section concludes with a formulation and discussion of an ultrafilter refinement of the

Furstenberg-Sárközy theorem on recurrence along polynomials, and a proof of a special case of this refinement.

Section 4 is devoted to ramifications of results brought in previous sections. Most of the discussion is devoted to polynomial ergodic theorems along IP-sets. In addition, the role of a polynomial refinement of the combinatorial Hales-Jewett theorem is emphasized. The flow of this discussion naturally leads to some open problems which are collected and commented on in Section 5.

I was fortunate to be a graduate student at the Hebrew University of Jerusalem at the time of the inception and early development of Ergodic Ramsey Theory. It is both my duty and pleasure to acknowledge the influence of and express my gratitude to Izzy Katznelson, Benji Weiss, and especially my Ph.D. thesis advisor, Hillel Furstenberg.

I wish to express my indebtedness to my friend and co-author Neil Hindman for many useful discussions of ultrafilter lore.

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1. Three main principles of Ramsey theory and its connection with the ergodic theory of multiple recurrence.

A mathematician, like a painter
or a poet, is a maker of patterns.

–G.H. Hardy, [Ha], p. 84.

Van der Waerden's Theorem, one of Khintchine's "Three Pearls of Number Theory" ([K1]), states that whenever the natural numbers are finitely partitioned (or, as it is customary to say, finitely colored), one of the cells of the partition contains arbitrarily long arithmetic progressions. One can reformulate van der Waerden's theorem in the following, "finitistic" form:

Theorem 1.1. For any natural numbers k and r there exists $N = N(k,r)$ such that whenever $m \geq N$ and $\{1,\cdots,m\} = \bigcup_{i=1}^{r} C_i$, one of C_i, $i = 1,\cdots,r$ contains a k-term arithmetic progression.

Exercise 1. Show the equivalence of van der Waerden's theorem and

Theorem 1.1.

Van der Waerden's theorem belongs to the vast variety of results which form the body of *Ramsey theory* and which have the following general form: If V is an infinite, "highly organized" structure (a semi-group, a vector space, a complete graph, etc.) then for any finite coloring of V there exist arbitrarily large (and sometimes even infinite) highly organized monochromatic substructures. In other words, the high level of organization cannot be destroyed by partitioning into finitely many pieces–one of these pieces will still be highly organized. To fit van der Waerden's theorem into this framework, let us call a subset of $\mathbf{Z}$ *a.p.-rich* if it contains arbitrarily long arithmetic progressions. Then van der Waerden's theorem can be reformulated in the following way:

Theorem 1.2. If $S \subset \mathbf{Z}$ is an a.p.-rich set and, for some $r \in \mathbf{N}$, $S = \bigcup_{i=1}^{r} C_i$, then one of C_i, $i = 1, \cdots, r$ is a.p.-rich.

(Since $\mathbf{Z}$ is a.p.-rich, van der Waerden's theorem is obviously a special case of Theorem 1.2. On the other hand, it is not hard to derive Theorem 1.2 from Theorem 1.1.)

We cannot resist the temptation to bring here two more equivalent forms of van der Waerden's theorem, each revealing still another of its facets.

Theorem 1.3. For any finite partition of $\mathbf{Z}$, one of the cells of the partition contains an affine image of any finite set. (An affine image of a set $F \subset \mathbf{Z}$ is any set of the form $a + bF = \{a + bx : x \in F\}$.)

Exercise 2. Show the equivalence of Theorems 1.3 and 1.1.

Theorem 1.4 (A special case of a theorem due to Furstenberg and Weiss, [FW1]). Suppose $k \in \mathbf{N}$ and $\epsilon > 0$. For any continuous self-mapping of a compact metric space (X, ρ), there exists $x \in X$ and $n \in \mathbf{N}$ such that $\rho(T^{in}x, x) < \epsilon$, $i = 1, \cdots, k$.

Theorem 1.3 shows that van der Waerden's theorem is actually a *geometric* rather than number theoretic fact. On the other hand, Theorem 1.4 establishes the seminal connection between partition theorems of van der Waerden type with *topological dynamics*–the link which proved to be extremely useful.

Another example of "unbreakable" structure is given by Hindman's finite sums theorem ([H2]). To formulate Hindman's theorem let us (following notation in [FW1]) call a set $S \subset \mathbf{N}$ an *IP-set* if it consists of an infinite sequence $(x_n)_{n=1}^{\infty} \subset \mathbf{N}$ together with all finite sums of the form $x_\alpha = \sum_{n\in\alpha} x_n$, where α ranges over the finite non-empty subsets of $\mathbf{N}$.

Theorem 1.5 (Hindman). If $E \subset \mathbf{N}$ is an IP-set, then for any finite coloring $E = \bigcup_{i=1}^{r} C_i$, one of C_i, $i = 1, \cdots, r$ contains an IP-set.

We shall return to Hindman's theorem in the discussions of Section 3. We refer the reader to [GRS] for many more examples illustrating the *first principle of Ramsey theory*–the *preservation of structure under finite partitions.*

After being convinced of the validity of this first principle of Ramsey theory, one is led to the next natural question: why is this so? What exactly is responsible for this stubborn tendency of highly organized infinite structures to preserve their (rightly interpreted) replicas in at least one cell of an arbitrary finite partition? The answer is, and this is the *second principle of Ramsey theory*, that there is always an appropriate notion of *largeness* which is behind the scenes and such that *any* large set contains the sought-after highly organized sub-structures. The only other requirement that the notion of largeness should satisfy is that if A is large and $A = \bigcup_{i=1}^r C_i$, then one of C_i, $1 = 1, \cdots, r$ is also large. It is the mathematician's task when dealing with this or that result of partition Ramsey theory to guess what the appropriate notion of largeness responsible for the truth of the proposition is. It is the almost intentional vagueness of the approach which allows one to obtain stronger and stronger theorems by modifying and playing with different notions of largeness. To illustrate this second principle of Ramsey theory we shall now give some examples.

Given a set $A \subset \mathbf{N}$, define its *upper density* $\overline{d}(A)$ by

$$\overline{d}(A) = \limsup_{N\to\infty} \frac{|A \cap \{1, \cdots, N\}|}{N}.$$

If the limit (rather than lim sup) exists, we say that A has density, and denote it by $d(A)$. Being of positive upper density is obviously a notion of largeness and it is natural to ask (as P. Erdös and P. Turán did in [ET]) whether this notion of largeness is responsible for the validity of van der Waerden's theorem. Namely, is it true that any set $A \subset \mathbf{N}$ of positive upper density is a.p.-rich?

The question turned out to be very hard. After some partial results were obtained in [Ro] and [Sz1], Szemerédi [Sz2] settled the Erdös-Turán conjecture affirmatively, thus providing a convenient sufficient condition for a set to be a.p.-rich.

Theorem 1.6 (Szemerédi, [Sz2]). Any set $E \subset \mathbf{N}$ having positive upper density is a.p.-rich.

Exercise 3. Derive from Theorem 1.6 the following finitistic version of it:

For any $\epsilon > 0$ and any $k \in \mathbf{N}$ there exists $N = N(\epsilon, k)$ such that if $m > N$ and $A \subset \{1, 2, \cdots, m\}$ satisfies $\frac{|A|}{m} > \epsilon$, then A contains a k-term arithmetic progression.

It follows from Exercise 3 that in the formulation of Theorem 1.6 a somewhat weaker notion of largeness would do, namely, the notion of *upper Banach density.* Given a set $E \subset \mathbf{Z}$ define its upper Banach density $d^*(E)$ by

$$d^*(E) = \limsup_{N-M\to\infty} \frac{|E \cap \{M, M+1, \cdots, N\}|}{N-M+1}.$$

It is the notion of positive upper Banach density and its natural extensions to $\mathbf{Z}^d$ and, indeed, to any countable *amenable* group which naturally participate in many questions and results of *density Ramsey theory.*

It is easy to check that for any $E \subset \mathbf{Z}$ and any $t \in \mathbf{Z}$ the set $E - t := \{x-t : x \in E\}$ satisfies $d^*(E-t) = d^*(E)$. This shift-invariance of the upper Banach density hints that there is a genuine measure preserving system behind any set $E \subset \mathbf{Z}$ with $d^*(E) > 0$. This is indeed so (see below). On the other hand, the notion of upper Banach density does not provide the right notion of largeness for results like Hindman's theorem. For example, the set $E = 2\mathbf{N}+1$ is large in the sense that $d^*(E) = \frac{1}{2}$ but obviously cannot contain any IP-set, or even any triple of the form $\{x, y, x+y\}$ (see also Exercise 8 in Section 3). We shall see in Section 3 that a notion of largeness relevant for Hindman's theorem is provided by idempotent ultrafilters in $\beta\mathbf{N}$, the Stone-Čech compactification of $\mathbf{N}$. This notion of largeness will also have a mild form of shift invariance which will allow us to prove Hindman's theorem by repeated utilization of a kind of Poincaré recurrence theorem adapted to the situation at hand.

Ergodic Ramsey Theory started with the publication of [F1], in which Furstenberg derived Szemerédi's theorem from a beautiful, far reaching extension of the classical Poincaré recurrence theorem, which corresponds to the case $k = 1$ in the following:

Theorem 1.7 (Furstenberg, [F1]). Let $(X, \mathcal{B}, \mu, T)$ be a probability measure preserving system. For any $k \in \mathbf{N}$ and for any $A \in \mathcal{B}$ with $\mu(A) > 0$ there exists $n \in \mathbf{N}$ such that

$$\mu(A \cap T^{-n}A \cap T^{-2n}A \cap \cdots \cap T^{-kn}A) > 0.$$

In order to derive Szemerédi's Theorem 1.6, Furstenberg introduced a *correspondence principle*, which provides the link between density Ramsey theory and ergodic theory.

Theorem 1.8 *Furstenberg's correspondence principle.* Given a set $E \subset \mathbf{Z}$ with $d^*(E) > 0$ there exists a probability measure preserving system $(X, \mathcal{B}, \mu, T)$ and a set $A \in \mathcal{B}$, $\mu(A) = d^*(E)$, such that for any $k \in \mathbf{N}$ and any $n_1, \cdots n_k \in \mathbf{Z}$ one has:

$$d^*\big(E \cap (E - n_1) \cap \cdots \cap (E - n_k)\big) \ge \mu(A \cap T^{-n_1}A \cap \cdots \cap T^{-n_k}A).$$

Since the set E contains a progression $\{x, x+n, \cdots, x+kn\}$ if and only if $E \cap (E-n) \cap \cdots \cap (E-kn) \neq \emptyset$, it is clear that Furstenberg's multiple recurrence Theorem 1.7 together with the correspondence principle imply Szemerédi's theorem. We remark that as a matter of fact, Theorem 1.7 follows from Szemerédi's theorem using fairly elementary arguments. Alternatively one can utilize the following refinement of the Poincaré recurrence theorem.

Theorem 1.9 ([B1]). For any probability measure preserving system $(X, \mathcal{B}, \mu, T)$ and any $A \in \mathcal{B}$ with $\mu(A) > 0$ there exists a sequence $E = (n_m)_{m=1}^{\infty}$ whose density exists and satisfies $d(E) \geq \mu(A)$ such that for any $m \in \mathbf{N}$

$$\mu(A \cap T^{-n_1}A \cap \cdots \cap T^{-n_m}A) > 0.$$

Van der Waerden's theorem has a natural multidimensional extension which is hinted at by the geometric formulation (Theorem 1.3).

Theorem 1.10 *Multidimensional van der Waerden theorem* (Gallai-Grünwald). For any finite coloring of $\mathbf{Z}^d$, $\mathbf{Z}^d = \bigcup_{i=1}^r C_i$, one of C_i, $i = 1, \cdots, r$ contains an affine image of any finite subset $F \subset \mathbf{Z}^d$. In other words, there exists i, $1 \leq i \leq r$, such that for any finite $F \subset \mathbf{Z}^d$, there exists $u \in \mathbf{Z}^d$ and $a \in \mathbf{N}$ such that $u + aF = \{u + ax : x \in F\} \subset C_i$.

Remark. An attribution of Theorem 1.10 to G. Grünwald is made in [Ra], p. 123. As far as we know, Grünwald never published his proof. He later changed his name to Gallai, to whom the result is attributed in [GRS].

In accordance with the second principle of Ramsey theory one should expect that Theorem 1.10 has a density version. This is indeed so and was proved in [FK1]. The multidimensional Szemerédi theorem established by Furstenberg and Katznelson there was the first in a chain of strong combinatorial results ([FK2], [FK4], [BL1]) which were achieved by means of ergodic theory and which so far have no conventional combinatorial proof.

Let us say that a set $S \subset \mathbf{Z}^k$ has positive upper Banach density if for some sequence of parallelepipeds $\Pi_n = [a_n^{(1)}, b_n^{(1)}] \times \cdots \times [a_n^{(k)}, b_n^{(k)}] \subset \mathbf{Z}^k$, $n \in \mathbf{N}$, with $b_n^{(i)} - a_n^{(i)} \to \infty$, $i = 1, \cdots, k$ one has:

$$\frac{|S \cap \Pi_n|}{|\Pi_n|} > \epsilon$$

for some $\epsilon > 0$.

The natural question now is whether it is true that any set of positive upper Banach density in $\mathbf{Z}^k$ contains an affine image of any finite set $F \subset \mathbf{Z}^k$. Furstenberg and Katznelson answered this question affirmatively by deducing the answer from the following generalization of Furstenberg's multiple recurrence theorem.

Theorem 1.11 ([FK1], Theorem A). Let $(X, \mathcal{B}, \mu)$ be a measure space with $\mu(X) < \infty$, let $T_1, \cdots, T_k$ be commuting measure preserving transformations of X and let $A \in \mathcal{B}$ with $\mu(A) > 0$. Then

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu(T_1^{-n}A \cap T_2^{-n}A \cap \cdots \cap T_k^{-n}A) > 0.$$

Corollary 1.12 ([FK1], Theorem B). Let $S \subset \mathbf{Z}^k$ be a subset with positive upper Banach density and let $F \subset \mathbf{Z}^k$ be a finite configuration. Then there exists a positive integer n and a vector $u \in \mathbf{Z}^k$ such that $u+nF \subset S$.

For the derivation of Corollary 1.12 from Theorem 1.11, the reader is referred to [F2], where the correspondence principle is spelled out for $\mathbf{Z}^k$.

The *third* and last *principle of Ramsey theory* which we want to discuss in this section is the following: *the sought-after configurations always to be found in large sets are abundant.* Abundance in our context means not only that the parameters describing the configurations form large sets in the space of parameters, but also that these parameters are nicely spread in all kinds of families of subsets of integers. Let us consider some examples. Take, for instance, Szemerédi's theorem. Let $E \subset \mathbf{Z}$ with $d^*(E) > 0$. For fixed k the progressions $\{x, x+d, \cdots, x+kd\} \subset E$ are naturally parametrized by pairs (x, d). Let us call a point $x \in E$ a (d, k)-starter if $\{x, x+d, \cdots, x+kd\} \subset E$ and a non-(d, k)-starter otherwise. One can show that for any k, "almost every" point of E is a (d, k)-starter for some d. In other words, for any fixed k, the set of (d, k)-starters in E has upper Banach density equal to $d^*(E)$.

Exercise 4. Show that the set of non-$(d, 2)$-starters of a set $E \subset \mathbf{Z}$ with $d^*(E) > 0$ may be infinite.

Let us turn now to a much more interesting set of those d which appear as differences of arithmetic progressions in E. One of the ways of measuring how well "spread" a subset of integers is, would be to see whether it has a nonempty intersection with different families of subsets of integers (analogy: a set $S \subset [0, 1]$ is dense if for any $0 \leq a < b \leq 1$, $S \cap (a, b) \neq \emptyset$). We shall need a few definitions. Given a countable abelian group G, a set $S \subset G$ is called *syndetic* if for some finite set $F \subset G$ one has: $S + F = \{x + y : x \in S, y \in F\} = G$. It is easy to see that a set $S \subset \mathbf{Z}$ is syndetic if and only if it has bounded gaps, namely intersects non-trivially any big enough interval. We note that any syndetic set $S \subset \mathbf{Z}$ is a.p.-rich. Indeed, as finitely many shifts of S cover $\mathbf{Z}$ completely, by van der Waerden's theorem one of these shifts is a.p.-rich, and as the property of a.p.-richness is clearly shift invariant, we see that S itself must have this property.

Following the terminology introduced in [F2], given a family $\mathcal{S}$ of subsets of $\mathbf{Z}$ let us call a set $E \subset \mathbf{Z}$ an $\mathcal{S}^*$-set if for any $S \in \mathcal{S}$, $E \cap S \neq \emptyset$. In particular, a set $E \subset \mathbf{Z}$ is an IP*-set if E has non-trivial intersection with any IP-set. It is not hard to see that any IP* set is syndetic. Indeed, if a set E was an IP*-set but not syndetic, its complement would contain a union of intervals $[a_n, b_n]$ with $b_n - a_n \to \infty$. One can easily show that any such union of intervals contains an IP-set which leads to contradiction with the assumed IP*-ness of E.

Exercise 5. Show that for any finite measure preserving system $(X, \mathcal{B}, \mu, T)$ and any $A \in \mathcal{B}$ with $\mu(A) > 0$, the set $\{n : \mu(A \cap T^{-n}A) > 0\}$ is an IP*-set.

On the other hand, it is easy to see that not every syndetic set is an IP*-set: take, for example, the odd integers.

Now, IP*-sets are large in a few different senses. Besides having positive lower density (the lower density of a set S is defined as $\liminf_{N\to\infty} \frac{|S\cap[1,\cdots,N]|}{N}$), they, for example, have a finite intersection property.

Lemma 1.13. If $S_1, S_2, \cdots, S_k$ are IP*-sets then $\bigcap_{i=1}^k S_i$ is also an IP*-set.

Proof. It is enough to prove the result for $k = 2$. Let E be an IP-set. Consider the following partition of E: $E = (E \cap S_1) \cup (E \cap S_1^c)$. By Hindman's theorem at least one of $E \cap S_1$, $E \cap S_1^c$ contains an IP-set E_1. Since S_1 is an IP*-set, $E_1 \cap S_1 \neq \emptyset$, hence $E_1 \subset E \cap S_1$. Also S_2 is an IP*-set, hence $E_1 \cap S_2 \neq \emptyset$, which implies that $E \cap (S_1 \cap S_2) \neq \emptyset$. As E was an arbitrary IP-set, the lemma is proved.

□

Given $E \subset \mathbf{Z}$ with $d^*(E) > 0$ let

$$R_k(E) = \big\{d \in \mathbf{Z} : \{x, x+d, \cdots, x+kd\} \subset E \text{ for some } x \in \mathbf{Z}\big\}.$$

The question of how well spread the sets $R_k(E)$ are in $\mathbf{Z}$ is interesting already for $k = 1$. The illustrative results about sets $R_1(E)$ which we collect here are special cases of sometimes very far reaching generalizations. Notice that $R_1(E) = E - E = \{x - y : x, y \in E\}$. It follows immediately from Exercise 5 via Furstenberg's correspondence principle that $R_1(E)$ is an IP*-set. We remark that this result has also a simple completely elementary proof: given an IP-set, generated, say, by $n_1, n_2, \cdots$, one considers the sets

$E_i = E - (n_1 + \cdots + n_i)$, $i = 1, 2, \cdots$ and observes that since $d^*(E) > 0$ we have, for some $1 \le i < j \le \frac{1}{d^*(E)} + 1$,

$$d^*(E_i \cap E_j) = d^*\Big(E \cap \big(E - (n_{i+1} + \cdots + n_j)\big)\Big) > 0.$$

This implies that the set of differences $E - E$ contains the element $n_{i+1} + \cdots + n_j$ from our IP-set.

We sketch now a curious application of this circle of ideas to the theory of almost periodic functions. For the sake of simplicity we shall deal only with functions on $\mathbf{Z}$, remarking that easy modifications of these arguments would apply to almost periodic functions on an arbitrary topological group. Recall that, according to H. Bohr ([Bo1]), a function $f : \mathbf{Z} \to \mathbf{C}$ is called almost periodic if for any $\epsilon > 0$ the set of "ϵ-periods",

$$E(\epsilon, f) = \{h \in \mathbf{Z} : |f(x+h) - f(x)| < \epsilon \text{ for all } x\}$$

is syndetic. Later Bogolioùboff, [Bo2], and Følner, [Fø] showed that the condition of syndeticity in the definition may be replaced by the weaker condition of positive upper Banach density. This result is contained in the following proposition:

Theorem 1.14. For a function $f : \mathbf{Z} \to \mathbf{C}$ the following conditions are equivalent:

(i) For any $\epsilon > 0$ the set $E(\epsilon, f)$ has positive upper Banach density.

(ii) For any $\epsilon > 0$ the set $E(\epsilon, f)$ is syndetic.

(iii) For any $\epsilon > 0$ the set $E(\epsilon, f)$ is an IP*-set.

Proof. It is enough to show that (i)→(iii). But this follows immediately from two facts:

(1) $E(\frac{\epsilon}{2}, f) - E(\frac{\epsilon}{2}, f) \subset E(\epsilon, f)$.

(2) If $d^*(E) > 0$ then $E - E$ is an IP*-set.

□

As a byproduct one obtains the following fact, which is not obvious from Bohr's definition (but is obvious from some other equivalent definitions of almost periodicity).

Corollary 1.15. If f, g are almost periodic functions then $f + g$ is also an almost periodic function.

Proof. Observe that $E(\frac{\epsilon}{2}, f) \cap E(\frac{\epsilon}{2}, g) \subset E(\epsilon, f + g)$. The result follows from Lemma 1.13.

□

Following Furstenberg ([F3]) let us call a set $R \subset \mathbf{Z}$ a *set of recurrence* if for any invertible finite measure preserving system $(X, \mathcal{B}, \mu, T)$ and any $A \in \mathcal{B}$ with $\mu(A) > 0$ there exists $n \in R$, $n \neq 0$, with $\mu(A \cap T^n A) > 0$.

Exercise 6. Show that the following are sets of recurrence:

(i) Any $E \subset \mathbf{Z}$ with $d^*(E) = 1$.

(ii) $a\mathbf{N} = \{an : n \in \mathbf{N}\}$, for any $0 \neq a \in \mathbf{Z}$.

(iii) $E - E$, for any infinite set $E \subset \mathbf{Z}$.

(iv) Any IP-set.

(v) Any set of the form $\bigcup_{n=1}^{\infty} \{a_n, 2a_n, \cdots, na_n\}$, $a_n \in \mathbf{N}$.

Denote by $\mathcal{R}$ the family of all sets of recurrence in $\mathbf{Z}$. According to our adopted conventions, a set E is $\mathcal{R}^*$ if it intersects nontrivially any set of recurrence. Similarly to IP-sets, sets of recurrence possess the Ramsey property: if a set of recurrence R is finitely partitioned, $R = \bigcup_{i=1}^{r} C_i$, then one of C_i, $i = 1, \cdots, r$ is itself a set of recurrence. To see this, assume that this is not true. So for a set of recurrence R and some partition $R = \bigcup_{i=1}^{r} C_i$ one can find measure preserving systems $(X_i, \mathcal{B}_i, \mu_i, T_i)$ and sets $A_i \in \mathcal{B}_i$, $i = 1, \cdots, r$, with $\mu_i(A_i) > 0$, such that $\mu_i(A_i \cap T_i^n A_i) = 0$ for all $n \in C_i$. Let $(X, \mathcal{B}, \mu, T)$ be the product system of $(X_i, \mathcal{B}_i, \mu_i, T_i)$, $i = 1, \cdots, r$ and take $A = A_1 \times \cdots \times A_r \in \mathcal{B}_1 \otimes \cdots \otimes \mathcal{B}_r$. Since R is a set of recurrence, there exists $n \in R$ such that $\mu(A \cap T^n A) > 0$, where $\mu = \mu_1 \times \cdots \times \mu_r$, $T = T_1 \times \cdots \times T_r$. This implies that for $i = 1, \cdots, r$, $\mu_i(A_i \cap T_i^n A_i) > 0$ which is a contradiction. The discussion above together with the fact that for any $E \subset \mathbf{Z}$ with $d^*(E) > 0$ the set $E - E$ is an $\mathcal{R}^*$-set imply the following combinatorial fact (cf. [F2], p. 75).

Theorem 1.16. Given sets $E_i \subset \mathbf{Z}$ with $d^*(E_i) > 0$, $i = 1, \cdots, k$, the set $D = (E_1 - E_1) \cap (E_2 - E_2) \cap \cdots \cap (E_k - E_k)$ is $\mathcal{R}^*$. In particular, D is IP* and hence syndetic.

The following fact, due independently to Furstenberg ([F2]) and Sárközy ([S]), provides an example of a set of recurrence of a quite different nature than those of Exercise 6.

Theorem 1.17. Assume that $p(t) \in \mathbf{Q}[t]$ with $p(\mathbf{Z}) \subset \mathbf{Z}$, $\deg p(t) > 0$, and $p(0) = 0$. The the set $\{p(n) : n \in \mathbf{N}\}$ is a set of recurrence.

For more examples and further discussion of sets of recurrence the reader is referred to [F3], [B1], [B2], [BHå], [Fo], and [M]. We comment now on some extensions of these results to multiple recurrence.

Given a finite invertible measure preserving system $(X, \mathcal{B}, \mu, T)$ and a set $A \in \mathcal{B}$ with $\mu(A) > 0$ consider the set

$$R_k(A) = \{n \in \mathbf{Z} : \mu(A \cap T^n A \cap \cdots \cap T^{kn} A) > 0\}.$$