Part One: Topology at infinity

1

End spaces

Throughout the book it is assumed that ANR spaces are locally compact, separable and metric, and that CW complexes are locally finite.

We start with the end space $e(W)$ of a space W, which is a homotopy theoretic model for the behaviour at ∞ of W. The homotopy type of $e(W)$ is determined by the proper homotopy type of W. The set of path components $\pi_0(e(W))$ is related to the number of ends of W, and the fundamental group $\pi_1(e(W))$ is related to the fundamental group at ∞ of W.

Definition 1.1 The *one-point compactification* of a topological space W is the compact topological space

$$W^\infty \;=\; W \cup \{\infty\} \;,$$

with open sets:

(i) $U \subset W^\infty$ for an open subset $U \subseteq W$,
(ii) $V \cup \{\infty\} \subseteq W^\infty$ for a subset $V \subseteq W$ such that $W \backslash V$ is compact. □

The topology at infinity of W is the topology of W^∞ at ∞.

Definition 1.2 The *end space* $e(W)$ of a space W is the space of paths

$$\omega \;:\; ([0,\infty],\{\infty\}) \longrightarrow (W^\infty,\{\infty\})$$

such that $\omega^{-1}(\infty) = \{\infty\}$, with the compact-open topology. □

The end space $e(W)$ is the *homotopy link* $\mathrm{holink}(W^\infty,\{\infty\})$ of $\{\infty\}$ in W^∞ in the sense of Quinn [116]. See 1.8 for the connection with the link in the sense of PL topology, and 12.11 for the general definition of the homotopy link.

We refer to Appendix B for a brief history of end spaces.

An element $\omega \in e(W)$ can also be viewed as a path $\omega : [0,\infty) \longrightarrow W$ such that $\omega(t)$ 'diverges to ∞' as $t \longrightarrow \infty$, meaning that for every compact subspace $K \subset W$ there exists $N > 0$ with $\omega([N,\infty)) \subset W \backslash K$.

Definition 1.3 (i) A map of spaces $f : V \longrightarrow W$ is *proper* if for each compact subspace $K \subseteq W$ the inverse image $f^{-1}(K) \subseteq V$ is compact. This is equivalent to the condition that f extends to a map $f^\infty : V^\infty \longrightarrow W^\infty$ of the one-point compactifications with $f^\infty(\infty) = \infty$.

(ii) A map $f : V \longrightarrow W$ is a *proper homotopy equivalence* if it is a proper map which is a homotopy equivalence in the proper category. □

We refer to Porter [111] for a survey of the applications of proper homotopy theory to ends. The end space $e(W)$ is called the 'Waldhausen boundary' of W in [111, p. 135].

An element $\omega \in e(W)$ is a proper map $\omega : [0,\infty) \longrightarrow W$, which is the same as a path in $\omega^\infty : [0,\infty] \longrightarrow W^\infty$ such that $\omega^\infty[0,\infty) \subseteq W$ and $\omega^\infty(\infty) = \infty$.

Example 1.4 (i) The end space of a compact space W is empty,

$$e(W) = \emptyset ,$$

since $W^\infty = W \cup \{\infty\}$ is disconnected and there are no paths $\omega^\infty : [0,\infty] \longrightarrow W^\infty$ from $\omega^\infty(0) \in W$ to $\omega^\infty(\infty) = \infty \in W^\infty$. The converse is false: the end space of $\mathbb{Z}$ is empty, yet $\mathbb{Z}$ is not compact.

(ii) Let T be a tree, and let $v \in T$ be a base vertex. A *simple edge path* in T is a sequence of adjoining edges $e_1, e_2, e_3, \dots$ (possibly infinite) without repetition. By the simplicial approximation theorem every proper map $\omega : [0,\infty) \longrightarrow T$ is proper homotopic to an infinite simple edge path starting at v. If T has at most a finite number of vertices of valency > 2 the end space $e(T)$ is homotopy equivalent to the discrete space with one point for each simple edge path of infinite length starting at $v \in T$.

(iii) The end space of $\mathbb{R}^+ = [0,\infty)$ is contractible,

$$e(\mathbb{R}^+) \simeq \{\mathrm{pt.}\} ,$$

corresponding to the unique infinite simple edge path starting at $0 \in \mathbb{R}^+$.

(iv) The end space of $\mathbb{R}$ is such that

$$e(\mathbb{R}) \simeq S^0 = \{+1, -1\} ,$$

corresponding to the two infinite simple edge paths starting at $0 \in \mathbb{R}$. □

In dealing with end spaces $e(W)$, we shall always assume that W is a locally compact Hausdorff space.

Remark 1.5 For any space W the evaluation map

$$p \; : \; e(W) \longrightarrow W \; ; \; \omega \longrightarrow \omega(0)$$

fits into a homotopy commutative square

$$\begin{array}{ccc} e(W) & \longrightarrow & \{\infty\} \\ {\scriptstyle p}\big\downarrow & & \big\downarrow \\ W & \xrightarrow{\;i\;} & W^{\infty} \end{array}$$

with $i : W \longrightarrow W^{\infty}$ the inclusion. The space W is 'forward tame' if and only if this square is a homotopy pushout rel $\{\infty\}$ – see Chapters 7, 12 for a more detailed discussion. □

Definition 1.6 Let $(K, L \subseteq K)$ be a pair of spaces. The space L is *collared* in K if the inclusion $L = L \times \{0\} \longrightarrow K$ extends to an open embedding $f : L \times [0, \infty) \longrightarrow K$. □

Proposition 1.7 *If (K, L) is a compact pair of spaces such that L is collared in K then the end space of the non-compact space*

$$W = K \backslash L$$

is such that there is defined a homotopy equivalence

$$L \longrightarrow e(W) \; ; \; x \longrightarrow \left(t \longrightarrow f\left(x, \frac{1}{1+t}\right)\right)$$

with $f : L \times [0, \infty) \longrightarrow K$ an open embedding extending the inclusion $L = L \times \{0\} \longrightarrow K$. □

In other words, if W is a non-compact space with a compactification K such that the boundary

$$\partial K = K \backslash W \subset K$$

is a compact subset which is collared in K then there is defined a homotopy equivalence

$$e(W) \simeq \partial K \; .$$

The homotopy theoretic 'space at infinity' $e(W)$ thus has the homotopy type of an actual space at infinity, provided ∂W is collared in the compactification K.

Example 1.8 (i) Let X be a compact polyhedron. For any $x \in X$ there exists a triangulation of X with x as a vertex, with the pair of compact spaces

$$(Y, Z) \;=\; (\mathrm{star}(x), \mathrm{link}(x))$$

such that $Y = x * Z$ is the cone on Z, and Z is collared in Y. (See Rourke and Sanderson [139] for the PL theory of stars and links.) The non-compact spaces

$$\begin{aligned} Y\backslash Z \;&=\; Z \times [0,\infty)/Z \times \{0\}\ , \\ W \;&=\; X\backslash\{x\} \;=\; \mathrm{cl}(X\backslash Y) \cup_{Z\times\{0\}} Z \times [0,\infty) \end{aligned}$$

have one-point compactifications

$$(Y\backslash Z)^\infty \;=\; Y/Z \quad , \quad W^\infty \;=\; X \quad (\infty = x)\ ,$$

with end spaces such that

$$e(Y\backslash Z) \;\simeq\; e(W) \;\simeq\; Z\ .$$

The homotopy link of $\{\infty\}$ in W^∞ is homotopy equivalent to the actual link of x in X.

(ii) Let $(M, \partial M)$ be a compact n-dimensional topological manifold with boundary. The boundary ∂M is collared in M. (In the topological category this was first proved by Brown [16]. See Conelly [31] for a more recent proof.) The interior of M is an open n-dimensional manifold

$$W \;=\; \mathrm{int}(M) \;=\; M\backslash\partial M$$

with an open embedding $f : \partial M \times [0,\infty) \longrightarrow M$ extending the inclusion $\partial M = \partial M \times \{0\} \longrightarrow M$. The end space of W is such that the map

$$g \;:\; \partial M \longrightarrow e(W)\ ;\ x \longrightarrow (t \longrightarrow f\Big(x, \frac{1}{t+1}\Big))$$

defines a homotopy equivalence, with the adjoint of g

$$\widehat{g} \;:\; \partial M \times [0,\infty) \longrightarrow W\ ;\ (x,t) \longrightarrow g(x)(t)$$

homotopic to f.

(iii) In view of (ii) a necessary condition for an open n-dimensional manifold W to be homeomorphic to the interior of a compact n-dimensional manifold with boundary is that the end space $e(W)$ have the homotopy type of a closed $(n-1)$-dimensional manifold. In Chapters 7, 8 we shall be studying geometric tameness conditions on W which ensure that $e(W)$ is at least a finitely dominated $(n-1)$-dimensional geometric Poincaré complex. □

The following result is a useful characterization of continuity for functions into an end space. It is based on elementary facts about the compact-open topology and proper maps.

Proposition 1.9 *For locally compact Hausdorff spaces X, W and a function $f : X \longrightarrow e(W)$, the following are equivalent:*

(i) *f is continuous,*
(ii) *the adjoint $\widehat{f} : X \times [0,\infty) \longrightarrow W; (x,t) \longrightarrow f(x)(t)$ is continuous, and for all compact subspaces $C \subseteq X$, $K \subseteq W$, there exists $N \geq 0$ such that $\widehat{f}(C \times [N,\infty)) \subseteq W \backslash K$,*
(iii) *for every compact subspace $C \subseteq X$, the restriction $\widehat{f}| : C \times [0,\infty) \longrightarrow W$ is a proper map.*

Proof (ii) $\Longleftrightarrow$ (iii) is obvious.
(i) $\Longrightarrow$ (iii) If f is continuous, so is the induced function

$$f^* \ : \ X \times [0,\infty] \longrightarrow W^\infty \ ; \ (x,t) \longrightarrow \begin{cases} f(x)(t) & \text{if } t < \infty \ , \\ \infty & \text{if } t = \infty \ . \end{cases}$$

Since $\widehat{f} = f^*|$, $\widehat{f}$ is continuous and $\widehat{f}| : C \times [0,\infty) \longrightarrow W$ is proper.
(iii) $\Longrightarrow$ (i) It suffices to show that the induced function $f^* : X \times [0,\infty] \longrightarrow W^\infty$ is continuous. It is clear that $f^*| : C \times [0,\infty] \longrightarrow W^\infty$ is continuous for each compact subspace $C \subseteq X$. The local compactness of X then implies that f^* is continuous. □

It follows that for a compact Hausdorff space X, a function $f : X \longrightarrow e(W)$ is continuous if and only if the adjoint $\widehat{f} : X \times [0,\infty) \longrightarrow W$ is a proper map. For non-compact X, W a constant map $X \longrightarrow e(W)$ is such that the adjoint $X \times [0,\infty) \longrightarrow W$ is not proper.

Proposition 1.10 *The end space defines a functor $e : W \longrightarrow e(W)$ from the category of topological spaces and proper maps to the category of topological spaces and all maps. A proper map $f : V \longrightarrow W$ induces a map*

$$e(f) \ : \ e(V) \longrightarrow e(W) \ ; \ \omega \longrightarrow f\omega \ ,$$

and a proper homotopy $f \simeq g : V \longrightarrow W$ induces a homotopy

$$e(f) \simeq e(g) \ : \ e(V) \longrightarrow e(W) \ . \qquad \square$$

A subspace $V \subseteq W$ is *cocompact* if the closure of $W \backslash V \subseteq W$ is compact. For a CW complex W a subcomplex $V \subseteq W$ is *cofinite* if it contains all but finitely many cells of W. A cofinite subcomplex is a cocompact subspace.

Definition 1.11 A space W is *σ-compact* if

$$W = \bigcup_{j=1}^{\infty} K_j$$

with each K_j compact and $K_j \subseteq K_{j+1}$. □

In particular, all the ANR's considered by us are σ-compact, since we are assuming that they are locally compact, separable and metric.

It follows from 1.10 that the homotopy type of $e(W)$ is determined by the proper homotopy type of W. A more general result will be established in 9.4 for a metric space W, that the homotopy type of $e(W)$ is determined by the 'proper homotopy type at ∞' of W. The inclusion of a closed cocompact subspace is a special case of a 'proper homotopy equivalence at ∞', and the following result will be used in the proof of 9.4:

Proposition 1.12 *If W is a σ-compact metric space and $u : V \longrightarrow W$ is the inclusion of a closed cocompact subspace then the inclusion of end spaces $e(u) : e(V) \longrightarrow e(W)$ is a homotopy equivalence.*

Proof Since W is a σ-compact metric space, W^{∞} and $e(W)$ are metrizable, and so $e(W)$ is paracompact. For each $\omega \in e(W)$ choose a number $t_\omega \in [0, \infty)$ such that

$$\omega([t_\omega, \infty)) \subseteq \mathrm{int}(V) .$$

Let $U(\omega)$ be an open neighbourhood of ω in $e(W)$ such that

$$\alpha([t_\omega, \infty)) \subseteq \mathrm{int}(V) \quad (\alpha \in U(\omega)) .$$

Let $\{U_i\}$ be a locally finite refinement of the covering $\{U(\omega) \,|\, \omega \in e(W)\}$ of $e(W)$, and let $\{\phi_i\}$ be a partition of unity subordinate to $\{U_i\}$. For each i choose $\omega_i \in e(W)$ such that $U_i \subseteq U(\omega_i)$, and let $t_i = t_{\omega_i}$. For each $\omega \in e(W)$ let

$$m_\omega = \min\{t_i \,|\, \phi_i(\omega) \neq 0\} .$$

Note that $\omega([m_\omega, \infty)) \subseteq \mathrm{int}V$ and $\sum\limits_i \phi_i(\omega) t_i \geq m_\omega$. The map

$$\begin{aligned} F \ : e(W) \times I &\longrightarrow e(W) ; \\ (\omega, t) &\longrightarrow (s \longrightarrow \omega((1-t)s + (\sum_i \phi_i(\omega) t_i + s)t)) \\ &(\omega \in e(W) \ , \ 0 \leq t \leq 1 \ , \ s \geq 0) \end{aligned}$$

is a deformation of $e(W)$ into $e(V)$ such that $F_t(e(V)) \subseteq e(V)$ for $0 \leq t \leq 1$. □

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Example 1.13 (i) The application of 1.12 to the inclusion

$$\{x \in \mathbb{R}^m \mid \|x\| \geq 1\} \;=\; S^{m-1} \times [1, \infty) \longrightarrow \mathbb{R}^m \quad (m \geq 1)\ ,$$

gives a homotopy equivalence

$$e(S^{m-1} \times [1, \infty)) \;\simeq\; e(\mathbb{R}^m)\ .$$

By 1.7 $e(S^{m-1} \times [1, \infty))$ is homotopy equivalent to S^{m-1}, so that

$$e(S^{m-1} \times [1, \infty)) \;\simeq\; e(\mathbb{R}^m) \;\simeq\; S^{m-1}\ .$$

(ii) Given a compact space K and an integer $m \geq 1$ let

$$W \;=\; K \times \mathbb{R}^m\ .$$

The one-point compactification $W^\infty = \Sigma^m K^\infty$ is the m-fold reduced suspension of $K^\infty = K \cup \{\text{pt.}\}$, and the end space is such that

$$e(W) \;\simeq\; K \times e(\mathbb{R}^m) \;\simeq\; K \times S^{m-1}\ . \qquad \square$$

In dealing with the number of ends of a space W we shall assume the following standing hypothesis for the rest of this chapter: *W is a locally compact, connected, locally connected Hausdorff space* (e.g. a locally finite connected CW complex).

In the literature the end space $e(W)$ has not played as central a role as the 'ends of W' or the 'number of ends of W'. Roughly, an end of W should correspond to a path component of $e(W)$. We now recall these classical notions and their relationship to $\pi_0(e(W))$.

Definition 1.14 (Milnor [100]) An *end* of a space W is a function

$$\epsilon : \{K \mid K \subseteq W \ \text{ is compact}\} \longrightarrow \{X \mid X \subseteq W\}\ ;\ K \longrightarrow \epsilon(K)$$

such that:

(i) $\epsilon(K)$ is a component of $W \backslash K$ for each K,
(ii) if $K \subseteq L$, then $\epsilon(L) \subseteq \epsilon(K)$.

A *neighbourhood* of ϵ is a connected open subset $U \subseteq W$ such that $U = \epsilon(K)$ for some non-empty compact $K \subseteq W$. $\square$

Remark 1.15 (i) For a σ-compact space W the definition of an end in 1.14 agrees with Definition 1 in the Introduction. A sequence $W \supseteq U_1 \supseteq U_2 \supseteq \dots$ of neighbourhoods of an end (in the sense of Definition 1 of the Introduction) such that $\bigcap\limits_{j=1}^{\infty} \text{cl}(U_j) = \emptyset$ determines an end ϵ of W (in the sense of 1.14) as follows: for a compact subspace $K \subseteq W$ choose j such that $U_j \cap K = \emptyset$ and let $\epsilon(K)$ be the component of $W \backslash K$ which contains U_j. On the other hand, if ϵ is an end of W and $W = \bigcup_{j=1}^{\infty} K_j$ with each K_j compact and $K_j \subseteq K_{j+1}$, then $\epsilon(K_j) = U_j$ defines a sequence of neighbourhoods of an end as above.

(ii) A subspace is *unbounded* if its closure is not compact. Note that if ϵ is an end of W, then $\epsilon(K)$ is unbounded for each compact subspace $K \subseteq W$. (Otherwise, $L = K \cup \mathrm{cl}(\epsilon(K))$ would be a compact subspace of W containing K, so $\epsilon(L) \subseteq \epsilon(K) \subseteq L$, contradicting $\epsilon(L) \subseteq W \backslash L$.) □

Definition 1.16 The *number of ends* of a locally finite *CW* complex W is the least upper bound of the number (which may be infinite) of infinite components of $W \backslash V$ for finite subcomplexes $V \subset W$. □

Example 1.17 (i) The real line $\mathbb{R}$ has exactly two ends.

(ii) The *dyadic tree* X is the tree embedded in $\mathbb{R}^2$ with each vertex of valency 3, with closure the union of X together with a disjoint Cantor set. The dyadic tree has an uncountable number of ends. See Diestel [37] for more information on ends of graphs. □

An alternative approach to the definition of an end is to focus attention on the number of ends of a space.

Definition 1.18 (Specker [151], Raymond [134]) The space W has *at least k ends* if there exists an open subspace $V \subseteq W$ with compact closure $\mathrm{cl}(V)$ such that $W \backslash \mathrm{cl}(V)$ has at least k unbounded components. The space W has *(exactly) k ends* if W has at least k ends but not at least $k+1$ ends. □

The point set conditions on W imply that if $V \subseteq W$ is an open subspace with compact closure, then $W \backslash \mathrm{cl}(V)$ has at most a finite number of unbounded components (see Hocking and Young [66, Theorem 3–9, p. 111]). If W has exactly k ends then there exists an open subspace $V \subseteq W$ with compact closure so that $W \backslash \mathrm{cl}(V)$ has exactly k unbounded components.

Proposition 1.19 *Let $k \geq 0$ be an integer.*

(i) *If W has at least k ends in the sense of Definition 1.14, then W has at least k ends in the sense of Definition 1.18.*

(ii) *If W is σ-compact and has at least k ends in the sense of Definition 1.18, then W has at least k ends in the sense of Definition 1.14.*

(iii) *For W σ-compact, W has exactly k ends in the sense of Definition 1.14 if and only if W has exactly k ends in the sense of Definition 1.18.*

Proof (i) Let $\epsilon_1, \ldots, \epsilon_k$ be distinct ends of W in the sense of 1.14. For $1 \leq i < j \leq k$, choose a compact subspace $H_{ij} \subseteq W$ such that $\epsilon_i(H_{ij}) \neq \epsilon_j(H_{ij})$.

It follows that $\epsilon_i(H_{ij}) \cap \epsilon_j(H_{ij}) = \emptyset$. Let

$$H = \bigcup_{1 \le i < j \le k} H_{ij} .$$

Since H is compact, there is an open subspace $V \subseteq W$ with compact closure such that $H \subseteq V$. Then $\epsilon_1(\mathrm{cl}(V)), \ldots, \epsilon_k(\mathrm{cl}(V))$ are unbounded components of $W \backslash \mathrm{cl}(V)$. Since $\epsilon_i(\mathrm{cl}(V)) \subseteq \epsilon_i(H_{ij})$, these are in fact k distinct components. Thus, W has at least k ends in the sense of 1.18.

(ii) We may assume that $k \ge 1$, so that W is non-compact. Note that if $K \subseteq W$ is a compact subspace, then $W \backslash K$ has at least one unbounded component. For if $V \subseteq W$ is an open subspace with compact closure such that $K \subseteq V$, then all but finitely many components of $W \backslash K$ are contained in V (see Hocking and Young [66, Theorem 3–9, p. 111]). It follows that one of those finitely many components of $W \backslash K$ must be unbounded.

Next, we shall show that if $K \subseteq W$ is a compact subspace and C is an unbounded component of $W \backslash K$, then there exists an end ϵ of W in the sense of 1.14 such that $\epsilon(K) = C$. For W can be written as

$$W = \bigcup_{j=0}^{\infty} K_j$$

with $K_0 = K$, each K_j compact and $K_j \subseteq K_{j+1}$. Define ϵ as follows. First, let $\epsilon(K) = \epsilon(K_0) = C$. Then, assuming $j \ge 1$ and that $\epsilon(K_{j-1})$ has been defined, define $\epsilon(K_j)$ to be one of the unbounded components of $\mathrm{cl}(\epsilon(K_{j-1})) \backslash K_j$ (which exists by the argument above). Finally, for an arbitrary compact subspace $H \subseteq W$, choose j such that $H \subseteq K_j$, and define $\epsilon(H)$ to be the component of $W \backslash H$ which contains $\epsilon(K_j)$. It is easy to verify that ϵ is an end of W in the sense of 1.14.

Since W has at least k ends in the sense of 1.18, there exists an open subspace $V \subset W$ with compact closure such that $W \backslash \mathrm{cl}(V)$ has at least k unbounded components, say $C_1, \ldots, C_k$. Then there exist ends $\epsilon_1, \ldots, \epsilon_k$ of W in the sense of 1.14 such that $\epsilon_j(\mathrm{cl}(V)) = C_j$ for $j = 1, \ldots, k$.

(iii) Immediate from (i) and (ii). □

If a space W is not assumed to be σ-compact, then we shall assume that an end of W refers to an end in the sense of 1.14 unless otherwise stated. Of course, such an end gives rise to an end in the sense of 1.18.

Proposition 1.20 *A connected space W with exactly k ends can be expressed as*

$$W = K \cup \bigcup_{j=1}^{k} W(j)$$

with $K \subseteq W$ a connected compact subspace, and each $W(j) \subseteq W$ a closed *connected subspace with exactly one end.*

Proof Let $V \subseteq W$ be an open subspace with compact closure such that $W \backslash \mathrm{cl}(V)$ has exactly k unbounded components, say $C_1, C_2, \dots, C_k$. Let

$$X = W \backslash \bigcup_{j=1}^{k} C_j = \mathrm{cl}(V) \cup \bigcup \{\text{all bounded components of } W \backslash \mathrm{cl}(V)\} .$$

Observe that X is compact. For if $\mathfrak{U}$ is a collection of open subsets of W which cover X, extract finitely many $U_1, U_2, \dots, U_n \in \mathfrak{U}$ such that $\mathrm{cl}(V) \subseteq \bigcup_{j=1}^{n} U_j$. Only finitely many of the components of $W \backslash \mathrm{cl}(V)$ are not contained in $\bigcup_{j=1}^{n} U_j$ (see Hocking and Young [66, Theorem 3–9, p. 111]). Let $D_1, D_2, \dots, D_m$ be the bounded components of $W \backslash \mathrm{cl}(V)$ not contained in $\bigcup_{j=1}^{n} U_j$. Then $\mathrm{cl}(D_j) \subseteq X$ is compact for each $j = 1, 2 \dots, m$. Thus, there exists a finite subcollection $\mathfrak{U}_j$ of $\mathfrak{U}$ which covers $\mathrm{cl}(D_j)$. Then

$$\{U_1, U_2, \dots, U_n\} \cup \mathfrak{U}_1 \cup \dots \cup \mathfrak{U}_m$$

is a finite subcollection of $\mathfrak{U}$ which covers X.

Now let $K \subseteq W$ be a compact connected subspace containing X (use Dugundji [38, page 254, exercise 2, section 6]) and let $W(j) = \mathrm{cl}(C_j)$ for $j = 1, 2, \dots, k$.

It only remains to see that each $W(i)$ has one end. Suppose on the contrary that ϵ_1 and ϵ_2 are distinct ends of $W(j)$. These ends induce ends $\widetilde{\epsilon}_1$, $\widetilde{\epsilon}_2$ of W by setting

$$\widetilde{\epsilon}_i(K) = \epsilon_i(K \cap W(j))$$

for $K \subseteq W$ and $i = 1, 2$. This shows that W has at least $k+1$ ends, a contradiction. □

Definition 1.21 The *set of ends* $\mathcal{E}_W$ of a space W is the set of ends of W in the sense of Definition 1.14. □

Proposition 1.22 (i) *The set of path components of the end space $e(W)$ is related to the set of ends of a space W by the map*

$$\eta_W : \pi_0(e(W)) \longrightarrow \mathcal{E}_W \; ; \; [\omega] \longrightarrow \epsilon_\omega$$

with $\epsilon_\omega(K)$ the component of $W \backslash K$ which contains $\omega([N, \infty))$, for any compact $K \subseteq W$.

(ii) *Given spaces X, Y, a closed cocompact subspace $U \subseteq X$ and a proper map $f : U \longrightarrow Y$ there is induced a map*

$$f_* : \mathcal{E}_X \longrightarrow \mathcal{E}_Y \; ; \; \epsilon \longrightarrow f_*(\epsilon)$$